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Hyperbolic Problems and Regularity Questions

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Preface

Research on hyperbolic problems and regularity questions developed so rapidly over the last years that a conference became necessary where recent progress could be discussed.

The conference covered a great variety of topics originating from nonlinear PDE, functional and applied analysis, physics, differential geometry, complex structures etc., that present a rich profile of studies in hyperbolic equations and related problems.

One objective was to bring together leading specialists from Europe, Asia and the United States and to discuss new challenges in this quickly developing field. Another goal of the conference was to exhibit the remarkable vitality and breadth of current activities in PDEs as well as the high scientific level of ongoing work in the area.

This volume collects polished versions of the lectures given at the conference. Readers will find inspiring contributions to the calculus of variations, differential geometry, the development of singularities, regularity theory, hydrodynamics, asymptotic behavior, among others, and will profit of new tools and ideas, new results, interesting points of view, and important problems waiting for their solution.

The conference was co-organized by the Department of Mathematics of the University of Ferrara and the Italian Ministry of University and Research. It took place from March 31 to April 3, 2004. We hope that this volume reflects the variety of topics discussed.

We are grateful to the members of the local Organizing Committee,

Marco Cappiello, University of Ferrara

Alessia Ascanelli, University of Ferrara

for their valuable help. We are happy to thank all the participants of this meeting for making it a success. Many thanks are due to Birkhäuser for constant encouragement and assistance.

Ferrara, April 2006

L. Zanghirati

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Hyperbolic Problems and Regularity Questions
Trends in Mathematics, 1–6

Some Applications of a Closed-form Solution for Compound Options of Order N

Rossella Agliardi

Keywords. Black-Scholes partial differential equation, multivariate normal integral correlation matrix.

1. Introduction

In this paper a closed formula for compound call options of order N is presented in the case of variable interest rate and volatility, thus generalizing the well-known Geske's expression for compound options. The result is obtained in a Black-Scholes framework, that is, solving N nested Cauchy problems for the Black-Scholes differential equation and using some properties of multivariate normal integrals in order to obtain a nice closed-form expression. Section 2 is devoted to sketch the proof. We refer to [2] for more details. In Section 3 our formula is applied to some important problems in Finance and Real Option Analysis.

2. Notation and main formula

In this section notation and assumptions are the same as in Black-Scholes environment. Here S will denote the current value of a stock and we will assume that S follows, as usual, the stochastic differential equation:

$$dS = \mu(t)Sdt + \sigma(t)SdW_t$$

where W_t is a standard Wiener process. Let $c_1(S,t)$ denote the value of a European call option on the stock, with exercise price X_1 and expiration date T_1, that is, such that $c_1(S,T_1) = \max(S - X_1, 0)$. As is well known, a closed-form solution for $c_1(S,t)$, $0 \le t \le T_1$, was derived in [3] and [5]. Let us now define inductively a sequence of call options (with value c_k) on the call option whose value is c_{k-1} and with exercise price X_k and expiration date T_k, where we assume $T_1 \ge T_2 \ge$

$\cdots \geq T_N$. The usual riskless hedging argument (see [3]) yields the following partial differential equations:

$$\frac{\partial c_k}{\partial t} = r(t)c_k - r(t)S\frac{\partial c_k}{\partial S} - \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 c_k}{\partial S^2} \qquad t \leq T_k,$$

for any $k = 1, \ldots, N$. The final condition is

$$c_k(c_{k-1}(S, T_k), T_k) = \max(c_{k-1}(S, T_k) - X_k, 0)$$

> *Our aim is to derive a valuation formula for the N-fold multicompound option, that is, for $c_N(S,t)$, $0 \leq t \leq T_N$.*

Let S_k^* denote the value of S such that $c_{k-1}(S, T_k) - X_k = 0$ if $k > 1$, and $S_1^* = X_1$. Let us define:

$$h_k(t) = \left(\ln\frac{S}{S_k^*} + \int_t^{T_k}\left(r(\tau) - \frac{\sigma^2(\tau)}{2}\right)d\tau\right) \Bigg/ \left(\int_t^{T_k}\sigma^2(\tau)d\tau\right)^{\frac{1}{2}} \tag{2.1}$$

and set

$$\rho_{ij}(t) = \left(\int_t^{T_j}\sigma^2(\tau)d\tau \Bigg/ \int_t^{T_i}\sigma^2(\tau)d\tau\right)^{\frac{1}{2}} \quad \textit{for} \quad 1 \leq i \leq j \leq k, t \leq T_k. \tag{2.2}$$

For any k, $1 \leq k \leq N$, let $\Xi_k^{(N)}(t)$ denote a k-dimension correlation matrix with typical element $\varrho_{ij}(t) = \rho_{N-k+i,N-k+j}(t)$. (Here we mean $\Xi_1^{(N)}(t) = 1$.) Let $N_k(h_k, \ldots, h_1; \Xi_k)$ denote the k-dimension multinormal cumulative distribution function, with upper limits of integration $h_1, \ldots, h_k$ and correlation matrix Ξ_k. The final result is that the value c_N of the compound option of order N is the following:

$$c_N(S,t) = SN_N\left(h_N(t) + \sqrt{\int_t^{T_N}\sigma^2(\tau)d\tau}, \ldots, h_1(t) + \sqrt{\int_t^{T_1}\sigma^2(\tau)d\tau}; \Xi_N^{(N)}(t)\right)$$
$$- \sum_{j=1}^{N} X_j e^{-\int_t^{T_j} r(\tau)d\tau} N_{N+1-j}\left(h_N(t), \ldots, h_j(t); \Xi_{N+1-j}^{(N)}(t)\right)$$

for any $0 \leq t \leq T_N$ and with the $h_j(t)$'s defined as above.

In what follows we just give a short outline of the proof: some additional details can be found in [2]. We argue inductively. The first step is to transform the Black-Scholes differential equation with $k = N$ into the heat equation

$$\partial_z Y_N = \partial_u^2 Y_N$$

by performing the following substitutions:

$$u = \ln(S_N^*/S) - \int_t^{T_N}\left(r(\tau) - \frac{1}{2}\sigma^2(\tau)\right)d\tau,$$

$$z = \frac{1}{2}\int_t^{T_N} \sigma^2(\tau)d\tau,$$

$$Y_N(u,z) = e^{\int_t^{T_N} r(\tau)d\tau} c_N(S,t).$$

Then we plug the expression for $Y_N(u,0)$ we can obtain by induction, into the solution of the Black-Scholes partial differential equation above, which is written in the form:

$$c_N(S,t) = e^{-\int_t^{T_N} r(\tau)d\tau} \int_{-\infty}^{+\infty} (\sqrt{4\pi z})^{-1} e^{-(u-\xi)^2/4z} Y_N(\xi,0)d\xi.$$

Thus, changing to variables $x = h_N(t) + \xi/\sqrt{2z}$, we have:

$$\begin{aligned}
c_N(S,t) = S &\int_{-\infty}^{h_N(t)+\sqrt{\int_t^{T_N}\sigma^2(\tau)d\tau}} (\sqrt{2\pi})^{-1} e^{-x^2/2} N_{N-1}((h_{N-1}(T_N) \\
&+ \sqrt{\int_{T_N}^{T_{N-1}} \sigma^2(\tau)d\tau} - x\rho_{N-1,N}(t)) \Big/ \sqrt{1-\rho^2_{N-1,N}(t)}, \ldots, (h_1(t) \\
&+ \sqrt{\int_{T_N}^{T_1} \sigma^2(\tau)d\tau} - x\rho_{1,N}(t)) \Big/ \sqrt{1-\rho^2_{1,N}(t)}; \Xi^{(N-1)}_{N-1}(T_N))dx \\
&- X_j e^{-\int_{T_N}^{T_j} r(\tau)d\tau} \int_{-\infty}^{h_N(t)} (\sqrt{2\pi})^{-1} e^{-x^2/2} \times \\
&\quad \times N_{N-j}((h_{N-1}(t) - x\rho_{N-1,N}(t)) \Big/ \sqrt{1-\rho^2_{N-1,N}(t)}, \ldots, (h_j(t) \\
&- x\rho_{j,N}(t)) \Big/ \sqrt{1-\rho^2_{j,N}(t)}; \Xi^{(N-1)}_{N-j}(T_N))dx \\
&+ X_N \int_{-\infty}^{0} (2\pi z)^{-1} e^{-x^2/2} dx
\end{aligned}$$

where $N_k(h_k,\ldots,h_1;\Xi_k)$ denotes the k-dimension multinormal cumulative distribution function, with upper limits of integration $h_1,\ldots,h_k$ and correlation matrix Ξ_k, and the entries $\varrho_{ij}(t)$ of $\Xi^{(N-1)}_k(t)$ are $\rho_{N-1-k+i,N-1-k+j}(t)$ for $i \le j$.

Since

$$\rho_{ij}(T_N) = (\rho_{ij}(t) - \rho_{iN}(t)\rho_{jN}(t)) \Big/ \sqrt{(1-\rho^2_{iN}(t))(1-\rho^2_{jN}(t))}$$

for $1 \le i < j \le N$, $t \le T_N$, we finally obtain the desired form from a general relationship between a k-dimension correlation matrix and its $(k-1)$-dimension partial correlation matrix (see [8], for example).

3. Some applications

The first application of our formula is to the valuation of default bonds. As Geske pointed out, mathematical expressions for the price of compound options can be applied to the valuation of risky coupon bonds. The equity of a firm that has coupon bonds outstanding can be valued as a compound option on the value of the firm and thus a valuation formula for a corporation's risky coupon bond with an arbitrary number of coupon payments can be obtained as a straightforward application of the result in Section 2. The formula we obtain in this section is a slight generalization of Geske's valuation formula for a corporate bond paying some coupons before the maturity date, assuming that the stockholders receive no dividends and might forfeit the firm when they are not able to pay the coupons. In this section we write down a valuation formula for a risky coupon bond with face value D and with N coupon payments. Let T be the maturity date of the bond, t_i be the coupon payment dates, $t_1 \prec \cdots \prec t_{N-1} \prec t_N = T$, and let X_i be the amount of the coupon owed at date t_i. Assume that the corporation has only common stock (with price S) and coupon bond outstanding and that it goes bankrupt at t_i if $S_{t_i} < X_i$. Let V be the value of the firm and assume that it follows a geometric Brownian process. Let $\overline{V}_i$ be the solution of $S(V, t_i) = X_i$ for $i < N$. If $V_{t_i} < \overline{V}_i$ then the corporation cannot pay the coupon and the bondholders receive V_{t_i}, while if $V_{t_i} > \overline{V}_i$ they get the coupon and hold the bond. On the other hand, the stock may be viewed as an N-fold compound call option on the value of the firm.

Thus the result we proved in the foregoing section yields:

$$
\begin{aligned}
S(V,0) = V N_N &\left(d_1 + \sqrt{\int_0^{t_1} \sigma^2(\tau)d\tau}, \ldots, d_N + \sqrt{\int_0^{t_N} \sigma^2(\tau)d\tau}; \Xi_N \right) \\
&- \sum_{i=1}^{N} X_i e^{-\int_0^{t_i} r(\tau)d\tau} N_i(d_1, \ldots, d_i; \Xi_i)) \\
&- D e^{-\int_0^{T} r(\tau)d\tau} N_N(d_1, \ldots, d_N; \Xi_N)
\end{aligned}
$$

where

$$
d_i(t) = \left(\ln \frac{V}{\overline{V}_i} + \int_0^{t_i} \left(r(\tau) - \frac{\sigma^2(\tau)}{2} \right) d\tau \right) \Bigg/ \left(\int_0^{t_i} \sigma^2(\tau) d\tau \right)^{\frac{1}{2}},
$$

with $\overline{V}_i$ defined above for $i < N$ and $\overline{V}_N = D + X_N$, and Ξ_k denotes a k-dimension correlation matrix with the ijth entry of the form

$$
\left(\int_0^{t_i} \sigma^2(\tau)d\tau \Bigg/ \int_0^{t_j} \sigma^2(\tau)d\tau \right)^{\frac{1}{2}}
$$

for $i \leq j$.

Thus we can finally write the price of the bond as follows:

$$
\begin{aligned}
B(V,0) = V &\left(1 - N_N\left(d_1 + \sqrt{\int_0^{t_1} \sigma^2(\tau)d\tau}, \ldots, d_N + \sqrt{\int_0^{t_N} \sigma^2(\tau)d\tau}; \Xi_N\right)\right) \\
&+ \sum_{i=1}^{N} X_i e^{-\int_0^{t_i} r(\tau)d\tau} N_i(d_1, \ldots, d_i; \Xi_i)) \\
&+ De^{-\int_0^{T} r(\tau)d\tau} N_N(d_1, \ldots, d_N; \Xi_N).
\end{aligned}
$$

Another important setting to which the mathematical approach of option pricing applies is real option analysis. For instance, an investment opportunity allowing management to expand the project's scale by a fraction α at time T, by making an investment outlay A, may be valued as a call option (see [7]). Indeed, if V_t denotes the gross project value at time t and it is assumed to follow a geometric Brownian motion, since just before the expiration of the option to expand, the investment opportunity's value is

$$V + \max(\alpha V - A, 0) = V + \alpha \max(V - A/\alpha, 0),$$

then the present additional value (given to the base-scale project by the option to expand) is $\alpha c_1(V,T,X)$, where $c_1(V,T,X)$ denotes the present value of a European call option with maturity date T and exercise price $X = A/\alpha$ and is given by Black-Scholes formula. However many investment projects consist in a combination of several real options, where earlier investment opportunities are prerequisites for others to follow, that is, the additional value of such opportunities may be valued as a compound option. For simplicity's sake we consider a project consisting in an option to expand by a fraction α_1 at time T_1 at an additional cost of A_1, followed by another option to expand by a fraction α_2 at time $T_2 \geqslant T_1$ at an additional cost of A_2. Let X_i denote A_i/α_i, $i = 1,2$ and let $c^{(2)}[T_1, \alpha_1, X_1; \alpha_2, T_2, X_2](V)$ denote the additional value of the whole project. Then a slight modification to the closed-form solution for the compound option of order 2 yields:

$$
\begin{aligned}
C^{(2)}\left[\alpha_1, T_1, X_1; \alpha_2, T_2, X_2\right](V) = \alpha_2 V N(\widetilde{h}_{X_2,T_2}) &- \alpha_2 X_2 e^{-\int_0^{T_2} r(\tau)d\tau} N(h_{X_2,T_2}) \\
&+ \alpha_1 V N(\widetilde{h}_{V*,T_1}) - \alpha_1 X_1 e^{-\int_0^{T_1} r(\tau)d\tau} N(h_{V*,T_1}) \\
&+ \alpha_1\alpha_2 V N_2(\widetilde{h}_{V*,T_1}, \widetilde{h}_{X_2,T_2}; \rho) \\
&- \alpha_1\alpha_2 X_2 e^{-\int_0^{T_2} r(\tau)d\tau} N_2(h_{V*,T_1}, h_{X_2,T_2}; \rho)
\end{aligned}
$$

where

$$h_{X,T} = \left(\ln\frac{V}{X} + \int_0^T \left(r(t) - \sigma^2(t)/2\right)dt\right) \Bigg/ \sqrt{\int_0^T \sigma^2(t)dt}\ ,$$

$$\widetilde{h}_{X,T} = h_{X,T} + \sqrt{\int_0^T \sigma^2(t)dt},$$

$$\rho = \sqrt{\int_0^{T_1} \sigma^2(t)dt} \Big/ \sqrt{\int_0^{T_2} \sigma^2(t)dt}$$

and $V*$ is such that $\alpha_2 c_1(V*, T_2 - T_1, X_2) + V* = X_1$.

Using this expression we can prove some properties which were firstly pointed out by Trigeorgis [6] by numerical valuation. For example, one can show that the super-additive effect holds, that is the combined value of two options to expand exceeds the sum of their individual value, since

$$C^{(2)}\left[\alpha_1, T_1, X_1; \alpha_2, T_2, X_2\right](V) \geqslant \alpha_1 c_1(V, T_1, X_1) + \alpha_2 c_1(V, T_2, X_2).$$

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Hyperbolic Problems and Regularity Questions
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Surjective Linear Partial Differential Operators with Variable Coefficients on Non-quasianalytic Classes of Roumieu Type

Angela A. Albanese

Abstract. Let P be a linear partial differential operator with variable coefficients in the Roumieu class $\mathcal{E}_{\{\omega\}}(\Omega)$. We prove that if P is $\{\omega\}$-hypoelliptic and has a $\{\omega\}$-fundamental kernel in Ω, then P is surjective on the space $\mathcal{E}_{\{\omega\}}(\Omega)$.

Mathematics Subject Classification (2000). Primary 35H10; Secondary 46F05, 47B38.

Keywords. Linear partial differential operators, fundamental kernel, surjectivity, ultradifferentiable functions.

1. Introduction

Surjectivity criteria for linear partial differential operators with constant coefficients have been obtained in most of the classical spaces of (ultradifferentiable) functions by several authors, see, e.g., [3, 4, 5, 6, 11, 15, 24, 25] and the references quoted therein. Hörmander [11] characterized the linear partial differential operators with constant coefficients which are surjective on all real-analytic functions on a given convex open set of $\mathbb{R}^N$. Sufficient conditions for the surjectivity of linear partial differential operators with constant coefficients on non-quasianalytic Gevrey classes $G^s(\Omega)$, $s > 1$, were first given by Cattabriga [5, 6] and Zampieri [24, 25], while a complete characterization on general ultradifferentiable classes $\mathcal{E}_{\{\omega\}}(\Omega)$ of Roumieu type was proved by Braun, Meise and Vogt [3, 4], and for any open sets by Langenbruch [15].

The aim of this paper is to consider the problem of the surjectivity for linear partial differential operators with variable coefficients on non-quasianalytic classes of Roumieu type. In particular, we show that the $\{\omega\}$-hypoellipticity and the existence of a $\{\omega\}$-fundamental kernel imply the surjectivity of linear partial differential operators with variable coefficients on the ultradifferentiable classes $\mathcal{E}_{\{\omega\}}(\Omega)$.

The result relies on an application of the projective limit functor of Palamodov [17] which shows that for each open set Ω of $\mathbb{R}^N$ a linear differential operator $P(x,D)$ is surjective on $\mathcal{E}_{\{\omega\}}(\Omega)$ if and only if $P(x,D)$ is locally surjective and $\mathrm{Proj}^1\mathcal{N}(\omega,P,\Omega)=0$, where $\mathcal{N}(\omega,P,\Omega)$ is a projective spectrum whose projective limit is $\operatorname{Ker} P(x,D)$. Applications are given to elliptic second order partial differential operators and to the Mizohata operator.

2. Preliminaries

In this section we fix notation and give some definitions and results which will be useful for the sequel.

Following Braun, Meise and Taylor [2], we introduce the classes of non-quasianalytic functions of Roumieu type.

Definition 2.1. A continuous increasing function $\omega\colon [0,\infty[\to [0,\infty[$ is called a *weight function* if it has the following properties:

- **(α):** there exists $K\geq 1$ with $\omega(2t)\leq K(1+\omega(t))$ for all $t\geq 0$,
- **(β):** $\int_0^\infty \frac{\omega(t)}{1+t^2}dt<\infty$,
- **(γ):** $\log t = o(\omega(t))$ for $t\to\infty$,
- **(δ):** $\varphi\colon t\to\omega(e^t)$ is convex on $\mathbb{R}$.

For a weight function ω we define $\tilde{\omega}\colon \mathbb{C}^N\to[0,\infty[$ by $\tilde{\omega}(z):=\omega(|z|)$ and again denote this function by ω. The *Young conjugate* $\varphi^*\colon [0,\infty[\to\mathbb{R}$ of φ is defined by

$$\varphi^*(y):=\sup_{x\geq 0}(xy-\varphi(x)).$$

Remark 2.2. (a) For each weight function ω we have $\lim_{t\to\infty}\omega(t)/t=0$.

(b) If ω and σ are two weight functions such that $\sigma(t)=\omega(t)$ for large $t>0$, then $\varphi_\sigma(y)=\varphi_\omega(y)$ for large y. It follows that all subsequent definitions coincide if ω is replaced by σ. They also coincide if ω is replaced by $\omega+c$, with $c>0$. Thus we can assume that $\omega(0)\geq 1$.

Example. The following functions $\omega\colon [0,\infty[\to[0,\infty[$ are examples of weight functions:

1. $\omega(t)=t^\alpha$, $0<\alpha<1$,
2. $\omega(t)=(\log(1+t))^\beta$, $\beta>1$,
3. $\omega(t)=\exp(\beta(\log(1+t))^\alpha))$, $0<\alpha<1$, $\beta>0$,
4. $\omega(t)=t(\log(e+t))^{-\beta}$, $\beta>1$.

We point out that for $\omega(t)=t^\alpha$ the classes of functions defined below coincide with the Gevrey class G^s for $s=1/\alpha$.

Definition 2.3. Let ω be a weight function.

(a) For a compact set K in $\mathbb{R}^N$ with $K=\overline{\mathring{K}}$ and $\mu>0$ let

$$\mathcal{E}_\omega(K,\mu):=\left\{f\in C^\infty(K):||f||_{K,\mu}:=\sup_{x\in K}\sup_{\alpha\in\mathbb{N}_0^N}|f^{(\alpha)}(x)|\exp\left(-\mu\varphi^*(|\alpha|/\mu)\right)<\infty\right\}$$

which is a Banach space endowed with the $|| \; ||_{K,\mu}$-topology.

(b) For a compact set K in $\mathbb{R}^N$ with $K = \overline{\overset{\circ}{K}}$ let

$$\begin{aligned}\mathcal{E}_{\{\omega\}}(K) &:= \left\{ f \in C^\infty(K) : \quad \text{there is } m \in \mathbb{N} \text{ with } ||f||_{K,1/m} < \infty \right\} \\ &= \operatorname*{ind}_{m\to\infty} \mathcal{E}_\omega(K, 1/m)\end{aligned}$$

which is the strong dual of a Fréchet Schwartz space (i.e., a (DFS)-space) if it is endowed with its natural inductive limit topology.

(c) For an open set Ω in $\mathbb{R}^N$ we define

$$\begin{aligned}\mathcal{E}_{\{\omega\}}(\Omega) &:= \left\{ f \in C^\infty(\Omega) : \quad \text{for each } K \subset\subset \Omega \text{ there is } m \in \mathbb{N} \; ||f||_{K,1/m} < \infty \right\} \\ &= \operatorname*{proj}_{K\subset\subset\Omega} \mathcal{E}_{\{\omega\}}(K)\end{aligned}$$

and we endow $\mathcal{E}_{\{\omega\}}(\Omega)$ with its natural projective topology. The elements of $\mathcal{E}_{\{\omega\}}(\Omega)$ are called *ω-ultradifferentiable functions of Roumieu type* on Ω. By [2], Proposition 4.9 $\mathcal{E}_{\{\omega\}}(\Omega)$ is a complete, nuclear and reflexive locally convex space. In particular, $\mathcal{E}_{\{\omega\}}(\Omega)$ is also an ultrabornological (hence barrelled and bornological) space as it follows from [20] (or see [23], Theorem 3.3.4) and [4], Lemma 1.8). We denote by $\mathcal{E}'_{\{\omega\}}(\Omega)$ the strong dual of $\mathcal{E}_{\{\omega\}}(\Omega)$.

(d) For a compact set K in $\mathbb{R}^N$ with $K = \overline{\overset{\circ}{K}}$ let

$$\mathcal{D}_{\{\omega\}}(K) := \left\{ f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^N) : \; \operatorname{supp} f \subset K \right\},$$

endowed with the induced topology. In [2], Remark 3.2-(1) and Corollary 3.6-(1), it is shown that $\mathcal{D}_{\{\omega\}}(K) \neq \{0\}$ is the strong dual of a Fréchet nuclear space (i.e., (DFN)-space). For an open set Ω in $\mathbb{R}^N$ let

$$\mathcal{D}_{\{\omega\}}(\Omega) := \operatorname*{ind}_{K\subset\subset\Omega} \mathcal{D}_{\{\omega\}}(K).$$

The elements of its strong dual $\mathcal{D}'_{\{\omega\}}(\Omega)$ are called *ω-ultradistributions of Roumieu type* on Ω.

We also recall that

$$\mathcal{E}_{\{\omega\}}(\Omega) = \operatorname*{proj}_{j\to\infty} \left(\mathcal{D}_{\{\omega\}}(K_j), \varphi^j_{j+1} \right),$$

where $K_1 \subset \overset{\circ}{K}_2 \subset K_2 \subset \cdots \subset \Omega$ is an exhaustation of Ω by compact sets, and the maps $\varphi^j_{j+1} \colon \mathcal{D}_{\{\omega\}}(K_{j+1}) \to \mathcal{D}_{\{\omega\}}(K_j)$ are defined by $\varphi^j_{j+1}(f) = \chi_j f$, with $\chi_j \in \mathcal{D}_{\{\omega\}}(K_j)$ satisfying $0 \le \chi_j \le 1$ and $\chi_j \equiv 1$ on K_{j-1} ([2], Lemma 4.5). In particular, for every compact subset K of Ω, $\mathcal{D}_{\{\omega\}}(K)$ carries the topology which is induced by $\mathcal{E}_{\{\omega\}}(\Omega)$ ([2], Lemma 4.6).

By (c) we also have that

$$\mathcal{E}_{\{\omega\}}(\Omega) = \operatorname*{proj}_{j\to\infty} (\mathcal{E}_{\{\omega\}}(K_j), \rho^j_{j+1}), \tag{2.1}$$

where the maps $\rho^j_{j+1} \colon \mathcal{E}_{\{\omega\}}(K_{j+1}) \to \mathcal{E}_{\{\omega\}}(K_j)$ are defined by $f \to \rho^j_{j+1}(f) := f_{|K_j}$.

Other notation is standard. We refer the reader for functional analysis, e.g., to [12, 19], and for the theory of linear partial differential operators to [10].

2.1. Partial differential operators

Let

$$P = P(x, D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$$

be a partial differential operator, where $D^\alpha = D_1^{\alpha_1} \dots D_N^{\alpha_N}$, $D_j = -i\partial/\partial x_j$, $|\alpha| = \alpha_1 + \dots + \alpha_N$ for any multi-index $\alpha \in \mathbb{N}_0^N$ and $m \in \mathbb{N}$.

Suppose that the coefficients a_α of P belong to $\mathcal{E}_{\{\omega\}}(\Omega)$, with Ω an open set of $\mathbb{R}^N$. Then P continuously maps every one of the spaces $\mathcal{E}_{\{\omega\}}(H)$, $\mathcal{E}'_{\{\omega\}}(H)$, $\mathcal{D}_{\{\omega\}}(H)$ and $\mathcal{D}'_{\{\omega\}}(H)$ into itself for every open or compact subset $H \subset \Omega$. We denote by $N(\omega, P, \Omega)$ (by $N(\omega, P, H)$, with $H \subset \Omega$, respectively) the kernel of P acting on $\mathcal{E}_{\{\omega\}}(\Omega)$ (on $\mathcal{E}_{\{\omega\}}(H)$ respectively).

To study the surjectivity of P on $\mathcal{E}_{\{\omega\}}(\Omega)$ we will use results on the projective limit functor. The theory of projective limit functor in the categories of vector spaces and locally convex spaces was introduced and developed by Palamodov [17]. Recent and significant progress of this theory is due to Vogt [20, 21] and Wengenroth [22, 23]. Following the presentation of Vogt [20], we recall the following.

Definition 2.4 (The projective limit functor). A sequence $\mathcal{X} = (X_n, i_{n+1}^n)_{n\in\mathbb{N}}$ of linear spaces X_n and linear maps $i_{n+1}^n \colon X_{n+1} \to X_n$ is said to be a projective spectrum. We define i_m^n by

$$i_n^n := I_{X_n} \quad \text{for all } n \in \mathbb{N}$$
$$i_m^n := i_{n+1}^n \circ \dots i_m^{m-1} \quad \text{if} \quad m > n.$$

For a projective spectrum $\mathcal{X} = (X_n, i_{n+1}^n)_{n\in\mathbb{N}}$ we define the linear spaces $\mathrm{Proj}^0\mathcal{X}$ and $\mathrm{Proj}^1\mathcal{X}$ by

$$X := \mathrm{Proj}^0\mathcal{X} := \Big\{(x_n)_{n\in\mathbb{N}} \in \prod X_n : \ i_{n+1}^n x_{n+1} = x_n \text{ for all } n \in \mathbb{N}\Big\}$$
$$\mathrm{Proj}^1\mathcal{X} := \Big(\prod X_n\Big) / B(\mathcal{X}),$$

where

$$B(\mathcal{X}) := \Big\{(a_n)_{n\in\mathbb{N}} \in \prod X_n : \ \exists (b_n)_{n\in\mathbb{N}} \in \prod X_n \, \forall n \in \mathbb{N} \; a_n = i_{n+1}^n(b_{n+1}) - b_n\Big\}.$$

We note that $\mathrm{Proj}^0\mathcal{X} = \mathrm{proj}_{\,n}(X_n, i_{n+1}^n)$.

Let ω be a weight function and Ω an open set of $\mathbb{R}^N$. Let $(K_n)_{n\in\mathbb{N}}$ be a sequence of compact subsets of Ω satisfying $\overline{\mathring{K}_n} = K_n$ and $K_n \subset \mathring{K}_{n+1}$ for all $n \in \mathbb{N}$. Then we denote by $\mathcal{E}_{\{\omega\}}^\Omega$ the projective spectrum $(\mathcal{E}_{\{\omega\}}(K_n), \rho_{n+1}^n)_{n\in\mathbb{N}}$, where $\rho_{n+1}^n \colon \mathcal{E}_{\{\omega\}}(K_{n+1}) \to \mathcal{E}_{\{\omega\}}(K_n)$, $f \to f_{|K_n}$. By [4], Lemma 1.8, we have that

$$\mathrm{Proj}^0\mathcal{E}_{\{\omega\}}^\Omega \simeq \mathcal{E}_{\{\omega\}}(\Omega) \quad \text{and} \quad \mathrm{Proj}^1\mathcal{E}_{\{\omega\}}^\Omega = 0. \tag{2.2}$$

Furthermore, let P be a linear partial differential operator defined in an open set $\Omega \subset \mathbb{R}^N$ with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. Then P induces a map

$$\mathcal{P} := (P_n^n)_{n\in\mathbb{N}} \colon \mathcal{E}_{\{\omega\}}^{\Omega} \to \mathcal{E}_{\{\omega\}}^{\Omega},$$

where $P_n^n \colon \mathcal{E}_{\{\omega\}}(K_n) \to \mathcal{E}_{\{\omega\}}(K_n)$, $f \to P_n^n f := Pf$.

Note that $\mathcal{N}(\omega, P, \Omega) := (N(P, \omega, K_n), \rho^n_{n+1|N(\omega,P,K_n)})_{n\in\mathbb{N}}$ is a projective spectrum, too. We denote by

$$\mathcal{J} := (\iota_n^n)_{n\in\mathbb{N}} \colon \mathcal{N}(\omega, P, \Omega) \to \mathcal{E}_{\{\omega\}}^{\Omega}$$

the natural inclusion, i.e., $\iota_n^n \colon N(P, \omega, K_n) \to \mathcal{E}_{\{\omega\}}(K_n)$, $f \to f$.

In order to state our main results, we introduce the following definition.

Definition 2.5. The linear partial differential operator P is said to be *locally surjective* in $\mathcal{E}_{\{\omega\}}(\Omega)$ if for each $n \in \mathbb{N}$ and $g \in \mathcal{E}_{\{\omega\}}(\Omega)$ there exists $f \in \mathcal{E}_{\{\omega\}}(\Omega)$ such that $P(f)_{|K_n} = g_{|K_n}$.

Locally surjective linear partial differential operators in the sense of Braun, Meise and Vogt [4], as defined above, correspond in the present context to the concept of semiglobally solvable operators as defined, e.g., in Trèves [19], Definition 38.2 page 392.

3. The results

Using the theory of projective limit functor, we obtain the following characterization for the surjectivity of linear partial differential operators with variable coefficients on non-quasianalytic classes of Roumieu type. We point out that its proof is along the lines of the one given in [4], Proposition 1.9, by Braun, Meise and Vogt to characterize the surjectivity of linear partial differential operators with constant coefficients on the same classes of ultradifferentiable functions.

Proposition 3.1. *Let P be a linear partial differential operator defined in an open set $\Omega \subset \mathbb{R}^N$ with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. Then $P \colon \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is surjective if and only if the following conditions are satisfied:*

(1) *P is locally surjective in $\mathcal{E}_{\{\omega\}}(\Omega)$,*
(2) $\mathrm{Proj}^1 \mathcal{N}(\omega, P, \Omega) = 0$.

Proof. Necessity: The surjectivity of P clearly implies that P is locally surjective in $\mathcal{E}_{\{\omega\}}(\Omega)$ and the following holds:

For each $n \in \mathbb{N}$ and each $g \in \mathcal{E}_{\{\omega\}}(K_{n+1})$ there exists $f \in \mathcal{E}_{\{\omega\}}(K_n)$ such that

$$P_n^n(f) = \rho^n_{n+1}(g). \tag{3.1}$$

Indeed, if we put $\varphi := \chi_{n+1} g \in \mathcal{E}_{\{\omega\}}(\Omega)$ (where $\chi_{n+1} \in \mathcal{D}_{\{\omega\}}(\Omega)$ satisfies $\chi_{n+1} \equiv 1$ on K_n, $0 \le \chi_{n+1} \le 1$, and $\mathrm{supp}\chi_{n+1} \subset K_{n+1}$), by the surjectivity of P there exists $\psi \in \mathcal{E}_{\{\omega\}}(\Omega)$ such that $P(\psi) = \varphi$, thereby implying that $f := \psi_{|K_n} \in \mathcal{E}_{\{\omega\}}(K_n)$ and $P_n^n(f) = (P\psi)_{|K_n} = \varphi_{K_n} = \rho^n_{n+1}(g)$.

Now (3.1) implies that

$$0 \to \mathcal{N}(\omega, P, \Omega) \xrightarrow{\mathcal{I}} \mathcal{E}^{\Omega}_{\{\omega\}} \xrightarrow{\mathcal{P}} \mathcal{E}^{\Omega}_{\{\omega\}} \to 0 \tag{3.2}$$

is an exact sequence of projective spectra (see [20], §1). Therefore, one of the main properties of the projective limit functor (see [17], page 542 or [20], Theorem 1.5) gives the following exact sequence of vector spaces

$$\begin{aligned} 0 \to \mathrm{Proj}^0\mathcal{N}(\omega, P, \Omega) &\xrightarrow{\mathcal{I}^0} \mathrm{Proj}^0\mathcal{E}^{\Omega}_{\{\omega\}} \xrightarrow{\mathcal{P}^0} \mathrm{Proj}^0\mathcal{E}^{\Omega}_{\{\omega\}} \\ \xrightarrow{\delta^*} \mathrm{Proj}^1\mathcal{N}(\omega, P, \Omega) &\xrightarrow{\mathcal{I}^1} \mathrm{Proj}^1\mathcal{E}^{\Omega}_{\{\omega\}} \xrightarrow{\mathcal{P}^1} \mathrm{Proj}^1\mathcal{E}^{\Omega}_{\{\omega\}} \to 0, \end{aligned} \tag{3.3}$$

where $\mathrm{Proj}^1\mathcal{E}^{\Omega}_{\{\omega\}} = 0$ by (2.2). By (2.2) we can also identify $\mathrm{Proj}^0\mathcal{E}^{\Omega}_{\{\omega\}}$ with $\mathcal{E}_{\{\omega\}}(\Omega)$. Then the map $\mathcal{P}^0$ corresponds to the differential operator $P\colon \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$. By (2.2) and the exactness of the sequence (3.3), this implies that $\delta^* = 0$ and

$$\mathrm{Proj}^1\mathcal{N}(\omega, P, \Omega) = \ker \mathcal{I}^1 = \operatorname{im} \delta^* = 0.$$

Sufficiency: Condition (1) implies that (3.1) holds and then the sequence in (3.2) is exact. Thus also the sequence in (3.3) is exact (see [17], page 542 or [20], Theorem 1.5), with $\mathrm{Proj}^1\mathcal{E}^{\Omega}_{\{\omega\}} = 0$. By (2) we have $\delta^* = 0$ and hence

$$\operatorname{im} \mathcal{P}^0 = \ker \delta^* = \mathrm{Proj}^0\mathcal{E}^{\Omega}_{\{\omega\}}.$$

Identifying $\mathrm{Proj}^0\mathcal{E}^{\Omega}_{\{\omega\}}$ with $\mathcal{E}_{\{\omega\}}(\Omega)$ and $\mathcal{P}^0$ with P again, we obtain that P is surjective. □

Since $\mathcal{N}(\omega, P, \Omega)$ is a projective spectrum of (DFN)-spaces (hence, of (DFS)-spaces) and $\mathrm{Proj}^0\mathcal{N}(\omega, P, \Omega) \simeq N(\omega, P, \Omega)$, we get from [20], Theorem 3.5 and [21]:

Corollary 3.2. *Let P be a linear partial differential operator defined in an open set $\Omega \subset \mathbb{R}^N$ with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. If $P\colon \mathcal{E}_{\{\omega\}}(\Omega) \to \mathcal{E}_{\{\omega\}}(\Omega)$ is surjective, then the kernel $N(\omega, P, \Omega)$ is ultrabornological and barrelled.*

Example. Consider the second order partial differential operator

$$P = \sum_{i,j=1}^{N} a_{ij}(x)\partial^2_{x_i x_j} + \sum_{i=1}^{N} b_i(x)\partial_{x_i} + c(x), \quad x \in \mathbb{R}^N.$$

Suppose that the following conditions are satisfied: $a_{ij} = a_{ji}$, $i, j = 1, \ldots, N$, all the coefficients a_{ij}, b_i and c are real-valued and belong to $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$, $c \le 0$, and the ellipticity condition

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \ge \nu(x)|\xi|^2, \quad x,\, \xi \in \mathbb{R}^N,$$

holds with $\inf_K \nu > 0$ for every compact subset K of $\mathbb{R}^N$.

Under these hypothesis, by the well-known theory of elliptic differential operators (see, e.g., [9], Theorems 6.13, 6.17) we have that for every $g \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ and $n \in \mathbb{N}$ there is a unique $u_n \in \mathcal{E}_{\{\omega\}}(B_n) \cap C(\overline{B_n})$ ($B_n := \{x \in \mathbb{R}^N : |x| < n\}$) satisfying the following Dirichlet problem

$$\begin{cases} Pu_n = g & \text{on } B_n \\ u_n = 0 & \text{on } \partial B_n \end{cases} \tag{3.4}$$

(as elliptic differential operators are of constant strength the Roumieu interior regularity follows, see, e.g., [10], Chapter VII, [18] and [8]). Thus P is locally surjective in $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$. Indeed, fixed any $g \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ and $n \in \mathbb{N}$, if we put $f := \chi_{n+1}u_{n+1}$ where u_{n+1} is the solution of the Dirichlet problem on B_{n+1} associated to P (here $\chi_{n+1} \in \mathcal{D}_{\{\omega\}}(\mathbb{R}^N)$ satisfies $\chi_{n+1} \equiv 1$ on B_n, $0 \le \chi_{n+1} \le 1$, and $\mathrm{supp}\chi_{n+1} \subset B_{n+1}$), we have that $f \in \mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$ and $P(f)_{|B_n} = P(u_{n+1})_{|B_n} = g_{|B_n}$.

On the other hand, by [1] $\mathrm{Ker}\, P$ is a nuclear Fréchet space, thereby implying that $\mathrm{Proj}^1 \mathcal{N}(\omega, P, \mathbb{R}^N) = 0$.

By Proposition 3.1 it follows that P is a surjective map on $\mathcal{E}_{\{\omega\}}(\mathbb{R}^N)$. $\square$

For the sequel we recall that we have the following canonical isomorphism

$$\mathcal{D}'_{\{\omega\}}(\Omega \times \Omega) \simeq L_\beta(\mathcal{D}_{\{\omega\}}(\Omega), \mathcal{D}'_{\{\omega\}}(\Omega)), \tag{3.5}$$

(where β denotes the topology of the uniform convergence on the bounded subsets of $\mathcal{D}_{\{\omega\}}(\Omega)$) which maps every $\{\omega\}$-ultradistribution $K(x, y)$ on $\Omega \times \Omega$ onto the linear continuous map $K : \mathcal{D}_{\{\omega\}}(\Omega) \to \mathcal{D}'_{\{\omega\}}(\Omega)$ defined by

$$\langle K(\psi), \varphi\rangle := \langle K, \varphi \otimes \psi\rangle \tag{3.6}$$

for all $\varphi,\ \psi \in \mathcal{D}_{\{\omega\}}(\Omega)$, where $(\varphi \otimes \psi)(x, y) := \varphi(x)\psi(y)$ for all $x,\ y \in \Omega$ (see [13], Theorem 2.3).

The $\{\omega\}$-ultradistribution $K(x, y)$ on $\Omega \times \Omega$ is said to be an *$\{\omega\}$-ultradifferentiable kernel* in Ω of the differential operator P if the corresponding map K defined in (3.6) satisfies

$$PK\varphi = \varphi$$

for all $\varphi \in \mathcal{D}_{\{\omega\}}(\Omega)$.

The operator P is said to be *$\{\omega\}$-hypoelliptic* in Ω if $Pu \in \mathcal{E}_{\{\omega\}}(\Omega')$ implies $u \in \mathcal{E}_{\{\omega\}}(\Omega')$ for every open set $\Omega' \subset \Omega$ and for every $u \in \mathcal{D}'_{\{\omega\}}(\Omega')$.

By Proposition 3.1 we are able to give the following sufficient condition for surjectivity of linear partial differential operator with variable coefficients on non-quasianalytic classes of Roumieu type.

Theorem 3.3. *Let P be a linear partial differential operator defined in an open set $\Omega \subset \mathbb{R}^N$ with coefficients in $\mathcal{E}_{\{\omega\}}(\Omega)$. If P is $\{\omega\}$-hypoelliptic in Ω and has a $\{\omega\}$-ultradifferentiable kernel K in Ω, then P is a surjective map from $\mathcal{E}_{\{\omega\}}(\Omega)$ onto $\mathcal{E}_{\{\omega\}}(\Omega)$.*

Proof. The $\{\omega\}$-hypoellipticity of P implies that the kernel $N(\omega, P, \Omega)$ is a nuclear Fréchet space by [1] and hence $\mathrm{Proj}^1\mathcal{N}(\omega, P, \Omega) = 0$, thereby condition (2) in Proposition 3.1 is satisfied.

By Proposition 3.1 it remains to show that P is locally surjective in $\mathcal{E}_{\{\omega\}}(\Omega)$. Fixed any $n \in \mathbb{N}$ and $g \in \mathcal{E}_{\{\omega\}}(\Omega)$, let us define $\varphi := \chi_{n+1}g \in \mathcal{D}_{\{\omega\}}(\Omega)$ (where $\chi_{n+1} \in \mathcal{D}_{\{\omega\}}(\Omega)$ satisfies $\chi_{n+1} \equiv 1$ on K_n, $0 \leq \chi_{n+1} \leq 1$, and $\mathrm{supp}\chi_{n+1} \subset K_{n+1}$). Then $K\varphi \in \mathcal{D}'_{\{\omega\}}(\Omega)$ and $PK\varphi = \varphi$ as K is a $\{\omega\}$-ultradifferentiable kernel in Ω of P. Now, by the $\{\omega\}$- hypoellipticity of P we have that $f := K\varphi \in \mathcal{E}_{\{\omega\}}(\Omega)$. On the other hand, we have

$$P(f)_{|K_n} = \varphi_{|K_n} = g_{|K_n}.$$

Thus, also condition (1) in Proposition 3.1 is satisfied. This completes the proof. $\square$

Example. Consider the Mizohata operator in $\mathbb{R}^2$:

$$P = \partial_{x_1} + ix_1^h\partial_{x_2},$$

where h is a fixed even integer. It is well known that a fundamental kernel in $\mathbb{R}^2$ of P and its transpose operator ${}^tP$ is given by

$$K(x, y) = \frac{1}{2\pi}\left(\frac{x_1^{h+1}}{h+1} + ix_2 - \frac{y_1^{h+1}}{h+1} - iy_2\right)^{-1}$$

and that $K(x, y)$ is an analytic function for $x \neq y$ (see, e.g., [18], Theorem 2.3.5). This clearly implies that P is $\{\omega\}$-hypoelliptic in $\mathbb{R}^2$ for every weight function ω.

By Theorem 3.3 we can conclude that the Mizohata operator P is a surjective map from $\mathcal{E}_{\{\omega\}}(\mathbb{R}^2)$ onto $\mathcal{E}_{\{\omega\}}(\mathbb{R}^2)$ for every weight function ω.

We point out that the surjectivity of the Mizohata operator P on the Gevrey classes $G^s(\mathbb{R}^2)$, $s \geq 1$, was already proved by Cattabriga and Zanghirati [7] via an analogue of the Cauchy–Kowalevsky theorem for ultradifferentiable functions due to Komatsu [14] and the G^s-hypoellipticiy of P on $\mathbb{R}^2$. Thus our result gives a new proof of the surjectivity of the Mizohata operator P on the Gevrey classes $G^s(\mathbb{R}^2)$ and at the same time permits us to extend it to every non-quasianalytic class of Roumieu type on $\mathbb{R}^2$.

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Hyperbolic Problems and Regularity Questions
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The Fundamental Solution for a Second Order Weakly Hyperbolic Cauchy problem

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Abstract. We construct the fundamental solution for a weakly hyperbolic operator satisfying an intermediate condition between effective hyperbolicity and the Levi condition. By the fundamental solution, we obtain the well-posedness in C of the Cauchy problem.

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Keywords. Weakly Hyperbolic Equations, Cauchy Problem.

1. Introduction and main results

Let us consider the Cauchy problem in $[0,T] \times \mathbf{R}^n$

$$\begin{cases} P(t,x,D_t,D_x)u(t,x) = 0, \quad (t,x) \in [0,T] \times \mathbf{R}^n\,, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.1}$$

for a second order operator

$$\begin{cases} P = D_t^2 - a(t,x,D_x) + b(t,x,D_x) + c(t,x), \\ a(t,x,\xi) = \displaystyle\sum_{i,j=1}^{n} a_{ij}(t,x)\xi_i\xi_j, \quad b(t,x,\xi) = \sum_{j=1}^{n} b_j(t,x)\xi_j, \end{cases} \tag{1.2}$$

$D = \frac{1}{\sqrt{-1}}\partial$, that satisfies the hyperbolicity condition

$$a(t,x,\xi) \geq 0, \quad t \in [0,T], x,\xi \in \mathbf{R}^n \tag{1.3}$$

with coefficients $a_{ij} \in C([0,T];\mathcal{B}^\infty)$, $b_j, c \in C^0([0,T];\mathcal{B}^\infty)$.

It is well known that well-posedness in C^∞ holds for an effectively hyperbolic operator and it is stable under any perturbation of the lower order terms $b(t,x,D_x), c(t,x)$. Otherwise, the first order term $b(t,x,\xi)$ has to satisfy Levi conditions. From [10], the condition

$$|\partial_x^\beta b(t,x,\xi)| \leq C_\beta \sqrt{a(t,x,\xi)}, \quad t \in [0,T], x,\xi \in \mathbf{R}^n\,,\ \beta \in \mathbf{Z}_+^n\,, \tag{1.4}$$

is sufficient in space dimension $n = 1$ assuming that the coefficients are analytic functions of the two variables t, x. The same holds true for any $n \geq 1$ with analytic coefficients $a_{ij}(t), b_j(t), c(t)$ depending only on the variable t, see [6].

An intermediate condition between effective hyperbolicity and (1.4) has been introduced in [5]. There the C^∞ well-posedness is proved taking C^∞ functions $a_{ij}(t), b_j(t), c(t)$ of the variable t and assuming that there is an integer $k \geq 2$ such that the symbols $a(t,\xi), b(t,\xi)$ satisfy

$$\sum_{j=0}^{k} |\partial_t^j a(t,\xi)| \neq 0, \quad |b(t,\xi)| \leq C a(t,\xi)^\gamma, \qquad t \in [0,T], |\xi| = 1, \tag{1.5}$$

with

$$\gamma + \frac{1}{k} \geq \frac{1}{2}. \tag{1.6}$$

Notice that for $a = a(t,\xi)$ independent of x, the effective hyperbolicity is equivalent to

$$a(t,\xi) = 0 \Rightarrow \partial_t^2 a(t,\xi) > 0 \quad t \in [0,T], |\xi| = 1,$$

that can be expressed also as follows

$$\sum_{j=0}^{2} |\partial_t^j a(t,\xi)| \neq 0, \quad t \in [0,T], |\xi| = 1.$$

This is in line with the fact that for $k = 2$ condition (1.6) is satisfied with $\gamma = 0$ (no Levi condition). On the other hand for $\gamma = 1/2$ one can take $k = \infty$, that means that under the Levi condition it is not necessary to assume that $a(t,\xi)$ has only finite order zeros. Furthermore (1.6) cannot be improved since the Cauchy problem for

$$P = D_t^2 - t^{2\ell} D_x^2 + t^\nu D_x \tag{1.7}$$

is well-posed in C^∞ if and only if

$$\nu \geq \ell - 1,$$

see [8].

The dependence on the space variable $x \in \mathbf{R}^n$, $n \geq 1$, of the lower order terms $b(t,x,D_x) + c(t,x)$ is allowed in [7]. There the C^∞ well-posedness is proved under the assumption

$$\sum_{j=0}^{k} |\partial_t^j a(t,\xi)| \neq 0, \quad |\partial_x^\beta b(t,x,\xi)| \leq C_\beta a(t,\xi)^\gamma, \tag{1.8}$$

$t \in [0,T]$, $x \in \mathbf{R}^n$, $|\xi| = 1$, $\beta \in \mathbf{Z}_+^n$, this time with the larger, for $k > 2$, value of γ

$$\gamma \geq \frac{1}{2} - \frac{1}{2(k-1)}. \tag{1.9}$$

Here we deal with operators satisfying such an intermediate condition between effective hyperbolicity and the Levi condition assuming that the principal term $a(t, x, D_x)$ in (1.2) is of the form

$$\begin{cases} a(t,x,\xi) = \alpha(t)Q(x,\xi), \quad \alpha \in C^\infty, \\ \alpha(t) \geq 0, \quad Q(x,\xi) = \displaystyle\sum_{i,j=1}^{n} q_{ij}(x)\xi_i\xi_j \geq q_0|\xi|^2, \quad q_0 > 0, \end{cases} \tag{1.10}$$

$t \in [0, T]$, $x, \xi \in \mathbf{R}^n$, and that there are $\gamma \geq 0$ and an integer $k \geq 2$ such that

$$\sum_{h=0}^{k} |\alpha^{(h)}(t)| \neq 0, \quad |\partial_x^\beta b_j(t,x)| \leq C_\beta \alpha(t)^\gamma, \tag{1.11}$$

$j = 1, \ldots, n$, $t \in [0, T]$, $x \in \mathbf{R}^n$, $\beta \in \mathbf{Z}^n_+$. For this class of operators, we prove the following:

Theorem 1.1. *Consider the Cauchy problem* (1.1) *for the operator* (1.2)*, under assumptions* (1.10)*,* (1.11)*,* (1.6)*. Then, problem* (1.1) *is well posed in* C^∞*.*

The result of Theorem 1.1 is optimal, since condition (1.6) is sharp to get C^∞ well-posedness also in the case of coefficients not depending on x, see [8]; in particular, it improves the results of [5] and [7] in the case of dimension $n = 1$.
A proof of Theorem 1.1 by means of an energy estimate with a finite loss of derivatives has already been given in [2]. The aim of this paper is to construct also the fundamental solution, and to use it to give a different proof of Theorem 1.1. This will allow us to investigate the propagation of the singularities of the solution of (1.1) in a future work.

We use a method developed by the second author in the study of degenerate hyperbolic Cauchy problems, see [3, 4, 1, 2], that consists of the following steps:

1. Factorization of the principal part of P by means of regularized characteristic roots.
2. For any given $f = f(t, x)$, reduction of the equation $Pu = f$ to an equivalent 2×2 system $LU = F$ with

$$L = D_t + \Lambda(t, x, D_x) + A(t, x, D_x), \tag{1.12}$$

where $\Lambda(t, x, \xi)$ is a real diagonal matrix of symbols of order 1 and the matrix $A(t, x, \xi)$ satisfies

$$\int_0^t |A(\tau, x, \xi)| d\tau \leq c_0 + \delta \log\langle\xi\rangle, \qquad c_0, \delta > 0, \ t \in [0, T], \tag{1.13}$$

denoting $\langle\xi\rangle = (1 + |\xi|^2)^{1/2}$.
3. Construction of the fundamental solution for the system L by means of multiproducts of Fourier integral operators, following [9]. Here the bound (1.13) leads to amplitudes of positive order δ, in line with the δ-loss of derivatives already observed in [2].

The reduction to a first order system is performed in Section 2. Section 3 is devoted to the construction of the fundamental solution of the system, see Theorem 3.1, and to the proof of Theorem 1.1, which will be a direct consequence of Theorem 3.1 and of the reduction performed in Section 2.

2. The reduction to a first order system

The aim of this section is to show that the Cauchy problem (1.1) can be reduced to a first order system $LU = F, U(0) = U_0$, with $L = D_t + \Lambda + A$ as in (1.12) and with (1.13) precised by the following estimate of the symbol A

$$\begin{cases} A \in L^1([0,T]; S^1), \\ |\partial_\xi^\alpha \partial_x^\beta A(t,x,\xi)| \le \rho_{\alpha\beta}(t,\xi)\langle\xi\rangle^{-|\alpha|}, \\ \rho_{\alpha\beta} \in C([0,T] \times \mathbf{R}^n), \\ \displaystyle\int_0^T \rho_{\alpha\beta}(t,\xi)dt \le \delta_{\alpha\beta}\log(1+\langle\xi\rangle), \quad \delta_{\alpha\beta} > 0. \end{cases} \tag{2.1}$$

In order to factorize the operator P, we define the approximated characteristic root

$$\tilde\lambda(t,x,\xi) = \sqrt{\alpha(t) + \langle\xi\rangle^{-2}}\sqrt{Q(x,\xi)}. \tag{2.2}$$

We need also to introduce the symbol

$$\omega(t,\xi) = \sqrt{1 + \alpha(t)\langle\xi\rangle^2},$$

so that

$$\tilde\lambda(t,x,\xi) = \langle\xi\rangle^{-1}\omega(t,\xi)\sqrt{Q(t,x,\xi)}.$$

Notice that

$$\omega \in C([0,T]; S^1), \quad \omega^{-1} \in C([0,T]; S^0), \quad \sqrt{\alpha}\omega^{-1} \in C([0,T]; S^{-1});$$

we assume (1.10) to have this good calculus for the approximated characteristic roots of P.

Furthermore, from (1.11), the symbols

$$\beta_0(t,\xi) := \alpha'(t)\langle\xi\rangle^2\omega^{-2}(t,\xi) = \frac{\alpha'(t)}{\alpha(t) + \langle\xi\rangle^{-2}}$$

and

$$\beta_1(t,x,\xi) := b(t,x,\xi)\omega^{-1}(t,\xi) = \frac{b(t,x,\xi)}{\langle\xi\rangle\sqrt{\alpha(t) + \langle\xi\rangle^{-2}}}$$

can be taken as entries of a matrix A satisfying (2.1). In fact, for any positive integer N, Lemma 1 in [6] implies that the function $\alpha^{1/N}$ is absolutely continuous, so we can write

$$\beta_0(t,\xi) = \frac{\alpha'(t)}{(\alpha(t) + \langle\xi\rangle^{-2})^{1-1/N}} \cdot \frac{1}{(\alpha(t) + \langle\xi\rangle^{-2})^{1/N}}$$

in order to get

$$\beta_0 \in L^1([0,T]; S^{2/N}).$$

For β_1, with $\gamma = 1/2 - 1/k$, we write

$$\beta_1(t,x,\xi) = \frac{b(t,x,\xi)}{\langle\xi\rangle(\alpha(t) + \langle\xi\rangle^{-2})^{\gamma}} \cdot \frac{1}{(\alpha(t) + \langle\xi\rangle^{-2})^{1/k}}$$

and we obtain $\beta_1 \in C([0,T]; S^{2/k})$, since

$$\frac{b(t,x,\xi)}{\langle\xi\rangle(\alpha(t) + \langle\xi\rangle^{-2})^{\gamma}}$$

is of order zero by assumption. Also the last condition in (2.1) is satisfied by the entries β_0, β_1 of A. In order to check this, one uses that $\alpha(t)$ has only isolated zeros of order less or equal than k. In a neighborhood of such a zero one just takes into account that

$$\int_0^T \frac{1}{(t^k + \langle\xi\rangle^{-2})^{1/k}} dt \le \int_0^{\langle\xi\rangle^{-2/k}} \frac{1}{\langle\xi\rangle^{-2/k}} dt + \int_{\langle\xi\rangle^{-2/k}}^T \frac{1}{t} dt = 1 + \log \frac{T}{\langle\xi\rangle^{-2/k}}$$

and that also α' vanishes at that point changing sign from minus to plus (cf. Lemma 1 and Lemma 2 in [5].)

So far, we can write the following factorization of P

$$P(t,x,D_t,D_x) = (D_t + \tilde\lambda(t,x,D_x))(D_t - \tilde\lambda(t,x,D_x)) + R_0(t,x,D_x)\omega(t,D_x) \quad (2.3)$$

with $R_0 \in L^1([0,T]; S^1)$ such that

$$\int_0^T \left|\partial_x^\beta \partial_\xi^\alpha R_0(t,x,\xi)\right| dt \le \delta_{\alpha\beta} \langle\xi\rangle^{-|\alpha|} \log(1 + \langle\xi\rangle). \quad (2.4)$$

Then, for a given scalar function $u(t,x)$, we define the vector $V = (v_0, v_1)$ by

$$\begin{cases} v_0 = \omega(t,D_x)u \\ v_1 = (D_t + \tilde\lambda(t,x,D_x))u. \end{cases} \quad (2.5)$$

The problem (1.1) for the operator (1.2), (1.10) is equivalent to the Cauchy problem

$$\begin{cases} L_1 V = 0, \\ V(0,x) = V_0, \end{cases} \quad (2.6)$$

for the first order system

$$L_1 = D_t + \begin{pmatrix} \tilde\lambda(t,x,D_x) & -\omega(t,D_x) \\ 0 & -\tilde\lambda(t,x,D_x) \end{pmatrix} + B_1(t,x,D_x)\alpha'(t)\langle D_x\rangle^2\omega^{-2}(t,D_x)$$

$$+C_1(t,x,D_x)b(t,x,D_x)\omega^{-1}(t,D_x) + R_1(t,x,D_x),$$

where $B_1(t,x,\xi), C_1(t,x,\xi), R_1(t,x,\xi) \in C^0([0,T]; S^0)$. The matrix

$$A_1 = B_1\alpha'(t)\langle\xi\rangle^2\omega^{-2} + C_1 b\omega^{-1} + R_1$$

satisfies (2.1) thanks to (2.4).

After a straightforward diagonalization, there is an elliptic symbol $M \in C([0,T];S^0)$ such that the Cauchy problem (2.6) is equivalent to the Cauchy problem

$$\begin{cases} LU = 0 \\ U(0,x) = U_0 \end{cases} \tag{2.7}$$

in the unknown $U = M(t,x,D_x)V$, where

$$L = D_t + \begin{pmatrix} \tilde{\lambda}(t,x,D_x) & 0 \\ 0 & -\tilde{\lambda}(t,x,D_x) \end{pmatrix} + B(t,x,D_x)\alpha'(t)\langle D_x\rangle^2\omega^{-2}(t,D_x) \\ + C(t,x,D_x)b(t,x,D_x)\omega^{-1}(t,D_x) + R(t,x,D_x)$$

with new 2×2 matrices $B, C, R \in C^0([0,T];S^0)$. The matrix

$$A = B\alpha'(t)\langle\xi\rangle^2\omega^{-2} + Cb\omega^{-1} + R$$

still satisfies (2.1).

3. The fundamental solution

We construct the fundamental solution of the operator L in (2.7) as a continuous family of operators $E(t,s)$ on $H^{-\infty}$, $t,s \in [0,T]$, such that

$$\begin{cases} LE(t,s) = 0, \\ E(s,s) = I, \end{cases}$$

provided that T is sufficiently small. We use the method of multi-products of Fourier integral operators by [9].

In the diagonal part of the symbol of L, we can put again the true roots $\pm\lambda = \pm\sqrt{\alpha Q}$ of P since

$$\int_0^T \left|\partial_x^\beta \partial_\xi^\alpha \left(\tilde{\lambda} - \lambda\right)\right| dt \le \delta_{\alpha\beta}\langle\xi\rangle^{-|\alpha|}\log(1+\langle\xi\rangle).$$

The homogeneous symbols $\pm\lambda$ in the variable ξ give canonical transformations $\mathcal{C}_\pm(t,s)$ in the cotangent bundle of $\mathbf{R}^n$

$$\mathcal{C}_\pm(t,s) : (y,\eta) \mapsto (x_\pm, \xi_\pm).$$

They are defined by the Hamilton-Jacobi equations

$$\begin{cases} \dfrac{dx_\pm}{dt} = \pm\nabla_\xi\lambda(t,x_\pm,\xi_\pm), \\ \dfrac{d\xi_\pm}{dt} = \mp\nabla_x\lambda(t,x_\pm,\xi_\pm), \\ (x_\pm,\xi_\pm)_{|t=s} = (y,\eta). \end{cases}$$

The corresponding phase-functions will be denoted by $\varphi_\pm(t,s;x,\eta)$. We need also the multi-phase-functions

$$\Psi^\nu(t,t_1,\ldots,t_\nu,s;x,\eta),$$

which are defined as the generating functions of the composed canonical transformations

$$\mathcal{C}^{(\nu)}(t,t_1,t_2,\ldots,t_\nu,s)=\mathcal{C}_{j_1}(t,t_1)\mathcal{C}_{j_2}(t_1,t_2)\cdots\mathcal{C}_{j_\nu}(t_{\nu-1},t_\nu)\mathcal{C}_{j_{\nu+1}}(t_\nu,s)$$

where each $\mathcal{C}_{j_k}$ is either $\mathcal{C}_+$ or $\mathcal{C}_-$.

For $\varphi(t,s;x,\eta)$ a real homogeneous phase function of order 1 and an amplitude $a(t,s;x,\eta)$ of order m, we denote by $A_\varphi(t,s;x,D_x)$ the Fourier integral operator from $H^{\mu+m}(\mathbf{R}^n)$ to $H^\mu(\mathbf{R}^n)$

$$A_\varphi(t,s;x,D_x)v(x)=(2\pi)^{-n}\int e^{i\varphi(t,s;x,\eta)}a(t,s;x,\eta)\widehat{v}(\eta)d\eta,$$

$\widehat{v}$ the Fourier transform of v. We use also the notation $a=\sigma(A_\varphi)$.

Let us consider the operators

$$I_\varphi=\begin{pmatrix} I_{\varphi_+}(t,s) & 0\\ 0 & I_{\varphi_-}(t,s)\end{pmatrix},\quad R_\varphi(t,s)=LI_\varphi(t,s),$$

and let us define the sequence

$$W_1(t,s)=-iR_\varphi(t,s),\quad W_{\nu+1}(t,s)=\int_s^t W_1(t,\tau)W_\nu(\tau,s)d\tau,\nu\geq 1.$$

We are going to prove the following:

Theorem 3.1. *For a sufficiently small T, there exists $\delta>0$ such that the fundamental solution of L, given by*

$$E(t,s)=I_\varphi(t,s)+\int_s^t I_\varphi(t,\tau)\sum_{\nu=1}^{\infty}W_\nu(\tau,s)d\tau,$$

is continuous from $H^{\mu+\delta}$ to H^μ for every μ.

Proof. Let us consider the sequence

$$E_N(t,s)=I_\varphi(t,s)+\int_s^t I_\varphi(t,\tau)\sum_{\nu=1}^{N}W_\nu(\tau,s)d\tau.$$

The entries of the matrix

$$W_\nu(t,s)=\int_s^t\cdots\int_s^{t_{\nu-2}}W_1(t,t_1)\cdots W_1(t_{\nu-1},s)dt_{\nu-1}\cdots dt_1$$

are Fourier integral operators with the multi-products $\Psi^\nu(t,t_1,\ldots,t_\nu,s;\,x,\eta)$ as phase-functions.

From (2.1) and the definition of W_1, we have

$$\begin{cases} |\partial_\xi^\alpha\partial_x^\beta\sigma(W_1(t,s))|\leq\rho_{\alpha\beta}(t,s,\xi)\langle\xi\rangle^{-|\alpha|},\\ \displaystyle\int_0^T\rho_{\alpha\beta}(t,s,\xi)dt\leq\delta_{\alpha\beta}\log(1+\langle\xi\rangle).\end{cases}\tag{3.1}$$

So, denoting

$$\rho_l(t,\xi) = \sup_{|\alpha+\beta|\le l,\ s\in[0,T]} \rho_{\alpha,\beta}(t,s,\xi),$$

from the calculus of multi-products of Fourier integral operators by [9], for every $l \in \mathbf{Z}_+$ there is an $l' \in \mathbf{Z}_+$ such that for $|\alpha+\beta| \le l, t_0 = t$, we have

$$|\partial_\xi^\alpha \partial_x^\beta \sigma(W_1(t,t_1)W_1(t_1,t_2)\cdots W_1(t_{\nu-1},s))| \le c_l^{\nu-1}\langle\xi\rangle^{-|\alpha|} \prod_{j=0}^{\nu-1} \rho_{l'}(t_j,\xi).$$

By symmetry and taking (3.1) into account, one obtains

$$\begin{aligned}\int_s^t \cdots \int_s^{t_{\nu-1}} \prod_{j=1}^{\nu} \rho_{l'}(t_j,\xi)dt_\nu \cdots dt_1 &= \frac{1}{\nu!}\int_s^t \cdots \int_s^t \prod_{j=1}^{\nu} \rho_{l'}(t_j,\xi)dt_\nu \cdots dt_1 \\ &\le \frac{1}{\nu!}\left(\delta_{l'}\log(1+\langle\xi\rangle)\right)^\nu, \quad \delta_{l'} = \sup_{|\alpha+\beta|\le l'} \delta_{\alpha,\beta},\end{aligned}$$

so, for $|\alpha+\beta| \le l$, we get

$$\left|\partial_\xi^\alpha \partial_x^\beta \sigma(E_N)(t,s;x,\xi)\right| \le C\langle\xi\rangle^{-|\alpha|} \sum_{\nu=0}^{N} \frac{\left(c_l\delta_{l'}\log(1+\langle\xi\rangle)\right)^\nu}{\nu!} \le C\langle\xi\rangle^{c_l\delta_{l'}-|\alpha|}.$$

Now, for any P with symbol in S^m, let us denote by l_0 the smallest integer such that

$$||Pu||_0 \le |P|_{l_0}^{(m)}||u||_m, \quad |P|_{l_0}^{(m)} := \sup_{|\alpha|+|\beta|\le l_0} \sup_{x,\xi} \langle\xi\rangle^{-m+|\alpha|}|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)|,$$

and define

$$\delta = c_{l_0}\delta_{l_0'};$$

then we have that E_N converges to a continuous operator E from H^δ to H^0. Since

$$L_\mu = \langle D_x\rangle^\mu L \langle D_x\rangle^{-\mu} = L + R_\mu$$

with R_μ of order zero, such an operator is continuous from $H^{\mu+\delta}$ to H^μ for every μ.

Proof of Theorem 1.1. Given any Cauchy data $U_0 \in H^{\mu+\delta}$, $F \in C([0,T];H^\mu)$, T sufficiently small and $\delta > 0$ as in Theorem 3.1, the Cauchy problem

$$\begin{cases} LU(t,x) = F(t,x) \\ U(0,x) = U_0 \end{cases}$$

admits a unique solution $U \in C([0,T];H^\mu)$ which is given by Duhamel's formula:

$$U(t) = E(t,0)U_0 + \int_0^t E(t,s)F(s)ds.$$

The equivalence between problems (1.1) and (2.7) gives then the existence of a unique solution $u \in C([0,T];H^{\mu+1}) \cap C^1([0,T];H^\mu)$ of (1.1). Thus, if we take C^∞ Cauchy data u_0, u_1, we get a unique C^∞ solution u, taking into account the finite speed of propagation of supports. □

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Hyperbolic Problems and Regularity Questions
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Pseudoholomorphic Discs Attached to Pseudoconcave Domains

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Abstract. We discuss almost complex perturbations of linear discs. We give precise estimates for the $(1,\alpha)$ norm of these deformations and for the dependence on parameters. In particular, we show how families of such discs give rise to local foliations of the space. Also, if Ω is a domain whose boundary is endowed with at least one negative eigenvector w_1 at 0 for the standard structure of $\mathbb{C}^n$, then small discs with velocity w_1 which are analytic for a C^1-perturbation of the structure, have boundary which is contained in Ω in a neighborhood of 0. In particular, if the almost complex structure is real analytic, almost holomorphic functions extend along the corresponding foliation of discs.

1. Preliminaries: the Cauchy singular integral

We organize here the classical results about singular integrals that we need for the construction of discs. Let Δ be the standard disc in $\mathbb{C}$, let z or τ be the variable in $\mathbb{C}$, and let z^1, z^2 be points in Δ.

Lemma 1.1. *We have*

$$\frac{1}{2\pi i}\iint_\Delta \frac{1}{(\tau-z^1)(\tau-z^2)}d\tau d\bar{\tau} = \frac{\overline{z^1-z^2}}{z^1-z^2}, \tag{1.1}$$

and

$$p.v.\frac{1}{2\pi i}\iint_\Delta \frac{1}{(\tau-z^1)^2}d\tau d\bar{\tau} = 0, \tag{1.2}$$

where "p.v." denotes the principal value.

Proof. For the first claim we remark that

$$\iint_\Delta \frac{1}{(\tau - z^1)(\tau - z^2)} d\tau d\bar\tau = -\int_{+\partial\Delta} \frac{\bar\tau}{(\tau - z^1)(\tau - z^2)} d\tau + \sum_{j=1,2} \int_{+\partial B(z^j,\eta)} \frac{\bar\tau}{(\tau - z^1)(\tau - z^2)} d\tau,$$

where $B(z^j, \eta)$, $j = 1, 2$, η small, denote the balls with centers z^j and radius η. We have

$$\frac{1}{2\pi i} \int_{+\partial\Delta} \frac{\bar\tau}{(\tau - z^1)(\tau - z^2)} d\tau = \sum \mathrm{Res}\big|_{0,z^1,z^2} \frac{1}{\tau(\tau - z^1)(\tau - z^2)} = 0, \tag{1.3}$$

$$\frac{1}{2\pi i} \int_{+\partial B(z^j,\eta)} \frac{\bar\tau}{(\tau - z^1)(\tau - z^2)} d\tau = \left((-1)^{j+1} \frac{\overline{z^j}}{z^1 - z^2} - \frac{\eta^2}{(z^1 - z^2)^2} \right) \tag{1.4}$$

for $j = 1, 2$. By taking summation over j and letting $\eta \to 0$ we get the proof of (1.1). As for the second claim, we notice that

$$\frac{1}{2\pi i} \int_{+\partial\Delta} \frac{\bar\tau}{(\tau - z^1)^2} d\tau = \sum \mathrm{Res}\big|_{0,z^1} \frac{1}{\tau(\tau - z^1)^2} = \frac{1}{(z^1)^2} - \frac{1}{(z^1)^2} = 0, \tag{1.5}$$

$$\int_{+\partial B(z^1,\eta)} \frac{\bar\tau}{(\tau - z^1)^2} d\tau = 0. \tag{1.6}$$

□

We denote by C^α, resp. $C^{1,\alpha}$, the space of functions which are α-Hölder continuous, respectively of differentiable functions whose derivative is C^α. For $f \in C^\alpha(\bar\Delta)$, we set $T_\Delta f(z) := \frac{1}{2\pi i} \iint_\Delta \frac{f(\tau)}{(\tau - z)} d\tau d\bar\tau$.

Proposition 1.2. *$T_\Delta f$ is differentiable and satisfies*

$$\partial_z T_\Delta f(z) = \frac{1}{2\pi i} \iint_\Delta \frac{f(\tau)}{(\tau - z)^2} d\tau d\bar\tau.$$

Proof. We have

$$\frac{T_\Delta f(z_o + \delta_z) - T_\Delta f(z_o)}{\delta_z} - \iint_\Delta \frac{f(\tau)}{(\tau - z_o)^2} d\tau d\bar\tau \tag{1.7}$$

$$= \iint_\Delta \frac{f(\tau)\delta_z}{(\tau - z_o)^2(\tau - z_o - \delta_z)} d\tau d\bar\tau.$$

We have to prove that (1.7) goes to 0 as $\delta_z \to 0$. We decompose the domain of integration as $\Delta = \Delta(z_o, t) \cup (\Delta \setminus \Delta(z_o, t))$ for $t = \delta_z^\gamma$ with $\gamma < \frac{1}{2}$. In $\Delta(z_o, t)$ we change variable $\xi := \tau - z_o$ and rewrite

$$\frac{\delta_z}{(\tau - z_o)^2(\tau - z_o - \delta_z)} = \frac{\delta_z}{\xi^2(\xi - \delta_z)} = -\xi^{-2} \left(1 + \frac{\xi}{\delta_z} + \cdots \right).$$

We may assume, without loss of generality, that $f(z_o) = 0$. We then rewrite, on account of the Hölder continuity of f:

$$\left| \frac{1}{2\pi i} \iint_{\Delta(z_o,t)} \cdot \right| \leq \frac{1}{\pi} \iint_{\Delta(0,t)} \frac{|\xi|^\alpha ||\delta_z|}{|\xi|^2 |\xi - \delta_z|} dV$$
$$= \frac{1}{\pi} \iint_{\Delta(0,t)} |\xi|^{-2+\alpha} (1 + \frac{|\xi|}{|\delta_z|} + \cdots) dV = O(|\delta_z|^{\alpha\gamma}),$$

where dV denotes the element of area in $\mathbb{C}$. On the other hand

$$\left| \iint_{\Delta \setminus \Delta(0,t)} \cdot \right| \leq c|\delta_z|^{1-2\gamma} \iint_\Delta \frac{|f|}{|\xi|} dV \to 0 \text{ for } \delta_z \to 0. \qquad \square$$

It is a little messy but classical that $T_\Delta f$ belongs in fact to $C^{1,\alpha}$ and that

$$f \to T_\Delta f,\ C^\alpha \mapsto C^{1,\alpha} \text{ is continuous.} \tag{1.8}$$

If we allow an arbitrarily small loss of regularity, that is $C^{1,\beta}$ instead of $C^{1,\alpha}$ for any $\beta < \alpha$, then the proof is straightforward. In fact, let us prove that for $f \in C^\alpha$ we have $\iint_\Delta \frac{f(\tau)}{(\tau - z^1)^2} d\tau d\bar{\tau} \in C^\beta$ for any $\beta < \alpha$. We have to estimate the difference of values taken at two different points z^1 and z^2. We assume that for one of the two points, say z^2, we have $1 - |z^2| > \frac{1}{2}|z^1 - z^2|$. (Otherwise, we triangulate over an additional point z^3 such that $|z^3 - z^1| = |z^2 - z^1|,\ |z^3 - z^2| = |z^2 - z^1|,\ 1 - |z^3| > \frac{1}{2}|z^1 - z^2|$.) We have

$$\begin{aligned} &\iint_\Delta \left(\frac{f(\tau)}{(\tau - z^1)^2} - \frac{f(\tau)}{(\tau - z^2)^2} \right) d\tau d\bar{\tau} \\ &= \iint_\Delta \frac{f(\tau) - f(z^1)}{(\tau - z^1)^2} d\tau d\bar{\tau} - \iint_\Delta \frac{f(\tau) - f(z^2)}{(\tau - z^2)^2} d\tau d\bar{\tau} \\ &= \iint_{-z^1+\Delta} \frac{f(z^1 + \tau) - f(z^1)}{\tau^2} d\tau d\bar{\tau} - \iint_{-z^2+\Delta} \frac{f(z^2 + \tau) - f(z^2)}{\tau^2} d\tau d\bar{\tau}, \end{aligned} \tag{1.9}$$

where the first equality is a consequence of Lemma 1.1. We rewrite the last line of (1.9) as

$$\begin{aligned} &\iint_{-z^1+\Delta} \frac{f(z^1 + \tau) - f(z^1) - (f(z^2 + \tau) - f(z^2))}{\tau^2} d\tau d\bar{\tau} \\ &+ \left(\iint_{-z^1+\Delta} \frac{f(z^2 + \tau) - f(z^2)}{\tau^2} - \iint_{-z^2+\Delta} \frac{f(z^2 + \tau) - f(z^2)}{\tau^2} d\tau d\bar{\tau} \right). \end{aligned} \tag{1.10}$$

We denote by (I) and (II) the two terms in (1.10). We remark that (II) can be estimated, after rescaling, by an integral over a rectangle $R = [s, t] \times [0, 1]$, with

$t = |z^1 - z^2|$ of type $\iint_R \frac{1}{(x_1^2+x_2^2)^{1-\frac{\alpha}{2}}}$. Hence

$$(II) \leq \int_s^t \frac{1}{x_1^{2-\alpha}} (\int_0^1 \frac{1}{(1+(\frac{x_2}{x_1})^2)^{1-\frac{\alpha}{2}}} x_1 d\frac{x_2}{x_1}) dx_1 \tag{1.11}$$
$$\leq c \int_s^t \frac{1}{x_1^{1-\alpha}} dx_1 \leq ct^\alpha.$$

Hence (II)$= O(|z^1 - z^2|^\alpha)$. As for (I) we fix ξ and define

$$g_\xi(z) := f(z+\xi) - f(z),$$

which is of course Hölder continuous. We have

$$|g_\xi(z^1) - g_\xi(z^2)| = \left|\left(f(z^1+\xi) - f(z^1)|\right) - \left(f(z^2+\xi) - f(z^2)\right)\right|$$
$$\leq c \inf(|z^1 - z^2|^\alpha, |\xi|^\alpha).$$

Hence

$$\left|\iint_{-z^1+\Delta} \frac{g_\xi}{\xi^2} dV\right| = \inf_\epsilon \left(\iint_{-z^1+\Delta\setminus\Delta(0,\epsilon)} \frac{|z^1-z^2|^\alpha}{|\xi|^2} dV + \iint_{\Delta(0,\epsilon)} |\xi|^{-2+\alpha} dV\right)$$
$$\leq c \inf_\epsilon \left(-\log \epsilon |z^1 - z^2|^\alpha + \epsilon^\alpha\right). \tag{1.12}$$

Since the infimum for the function $-\log \epsilon |z^1 - z^2|^\alpha + \epsilon^\alpha$ is attained for $\epsilon = \alpha^{\frac{1}{\alpha}} |z^1 - z^2|$, we can conclude

$$(I) \leq c(-\log(|z^1 - z^2|)|z^1 - z^2|^\alpha + |z^1 - z^2|^\alpha),$$

for a suitable constant c.

2. Analytic discs in almost complex structures

Let X be a real manifold equipped with an almost complex structure J that is an antiinvolution $J^2 = -\mathrm{id}$. There are then two bundles $T^{(0,1)}X$ and $T^{(1,0)}X$ in $\mathbb{C} \otimes TX$, the eigenspaces of $-i$ and i for $J^{\mathbb{C}}$ respectively. We have that $\mathbb{C} \otimes TX$ is the direct sum of the bundles $T^{(1,0)}X \oplus T^{(1,0)}X$ but the two bundles are not involutive, in general: involutivity characterizes complex structures according to the celebrated theorem by Newlander-Nirenberg. For a submanifold $M \subset X$, we set $T^{\mathbb{C}}M = TM \cap JTM$, $T^{(0,1)}M = T^{(1,0)}X \cap (\mathbb{C} \otimes TM)$, $T^{(1,0)}M = T^{(0,1)}X \cap (\mathbb{C} \otimes TM)$. These three distributions of vector fields are isomorphic and, in case they have constant rank, M is said CR. CR functions are the (continuous) solutions f of the equations $\bar{L}f = 0\ \forall \bar{L} \in T^{(0,1)}M$. For the complexified dual bundle we have a decomposition $\mathbb{C} \otimes T^*X = (T^{(1,0)}X)^* \oplus (T^{(0,1)}X)^*$ the sum of the forms which annihilate $T^{(0,1)}X$ and $T^{(1,0)}X$ respectively. $T^*_M X$ will denote the bundle of real forms which are purely imaginary over TM. ∂ and $\bar{\partial}$ will denote the components of d in $(T^{(1,0)}X)^*$ and $(T^{(0,1)}X)^*$ resp. We assume now that M is generic that is $TX = TM + JTM$, which is always the case when M has codimension 1. In this situation J provides an identification $\frac{TM}{T^{\mathbb{C}}M} \to T_M X$. By the aid of this identification, we define the Levi form of M at a point z in M by setting, for any $u \in T_z^{\mathbb{C}}M$:

$$\mathcal{L}_M(u,\bar{u}) := \frac{1}{2i}J[L,\bar{L}] \text{ in } T_MX,$$

where L is any section of $T^{(0,1)}M$ such that $L(z) = u$. In T_M^*X, identified to $\mathbb{R}^l$ by a choice of a basis $\partial r_1, \dots, \partial r_l$ where $r_j = 0$ is a system of independent equations for M, we have, on account of Cartan's formula:

$$\frac{1}{2i}\left(J^* dr_j[L,\bar{L}]\right)_j = \frac{1}{2i}\left((-\partial\bar{\partial} + \bar{\partial}\partial + \partial^2 - \bar{\partial}^2)r_j\, u \wedge \bar{u}\right)_j.$$

(This is called the "extrinsic" Levi form.) We select now a basis of $T^{(0,1)}X$ vector fields, say

$$\bar{L}_j = \partial_{\bar{z}_j} + \sum_k \lambda_{kj}\partial_{z_k}. \tag{2.1}$$

We may suppose that

$$\lambda_{kj}|_0 = 0, \qquad \text{and } X \text{ is a neighborhood of } 0 \text{ in } \mathbb{C}^n. \tag{2.2}$$

(We also remark that if $\partial_z\lambda|_{z_o} = 0$, then $\mathcal{L}_M(z_o) \lrcorner dr = \operatorname{Im}\bar{\partial}\partial r(z_o)$, the Levi form for the standard structure.)

An analytic disc A in X is a holomorphic map from the standard disc $\Delta \subset \mathbb{C}$:

$$A : \Delta \to X \text{ such that } A_*(\partial_{\bar{\tau}}) \in T^{(0,1)}X.$$

We will denote by A both the parametrization and the image set. Given an analytic disc $\tau \to w(\tau)$ for the standard structure, we are interested on its perturbations $w(\tau) + \epsilon(\tau)$ which are almost complex. This leads to the equation

$$\partial_{\bar{\tau}}\epsilon - \lambda(w+\epsilon)\partial_{\bar{\tau}}(\overline{w+\epsilon}) = 0,$$

that we can also put into the integral form

$$\epsilon(z) - \frac{1}{2\pi i}\iint_\Delta \frac{\lambda(w+\epsilon)\partial_{\bar{\tau}}\overline{w+\epsilon}}{\tau - z}d\tau d\bar{\tau} = 0 \ \ \forall z \in \Delta. \tag{2.3}$$

We denote this equation by $F = 0$. It is convenient to rewrite our standard disc as $w_o + w(\tau)$ with the normalization $w(0) = 0$. We consider

$$F \,:\, \mathbb{C}^n \times C^{1,\alpha}(\Delta) \times C^{1,\alpha}(\Delta) \to C^{1,\alpha}(\Delta), \tag{2.4}$$

$$F \,:\, (w_o, w(z), \epsilon(z)) \to \epsilon(z) - \frac{1}{2\pi i}\iint_\Delta \frac{\lambda(w_o + w(\tau) + \epsilon(\tau))\partial_{\bar{\tau}}\overline{(w(\tau)+\epsilon(\tau))}}{\tau - z}d\tau d\bar{\tau}. \tag{2.5}$$

Proposition 2.1. *F is C^1 as application between functional spaces and $\partial_\epsilon F$ is close to the identity.*

Proof. It is obvious that

$$\begin{aligned}&(w_o, w, \epsilon) \to \lambda(w_o + w + \epsilon)\partial_\tau(w_o + w + \epsilon),\\ &C^{1,\alpha}(\Delta) \to C^\alpha(\Delta),\end{aligned}$$

is continuously differentiable. On the other hand, by Proposition 1.2 and by (1.8), we have that

$$\lambda(w_o+w+\epsilon)\partial_\tau(w_o+w+\epsilon)\to\frac{1}{2\pi i}\iint_\Delta\frac{\lambda(w_o+w+\epsilon)\partial_{\bar\tau}(\overline{w_o+w+\epsilon})}{(\tau-z)}d\tau d\bar\tau,$$
$$C^\alpha(\Delta)\to C^{1,\alpha}(\Delta),$$

is continuous. □

Remark 2.2. It is immediate to check that the differential of F is described by

$$F'\ :\ (\dot w_o,\dot w,\dot\epsilon)\mapsto\dot\epsilon-\iint_\Delta\frac{\lambda'(\dot w_o+\dot w+\dot\epsilon)\partial_{\bar\tau}(\overline{w_o+w+\epsilon})+\lambda\partial_{\bar\tau}(\overline{\dot w+\dot\epsilon})}{(\tau-z)}d\tau d\bar\tau.$$

(Here the "prime" denotes the Jacobian.) In particular, $\partial_\epsilon F:\ C^{1,\alpha}(\Delta)\to C^{1,\alpha}(\Delta)$ is invertible in a neighborhood of 0.

It follows

Theorem 2.3. *For any* $(w_o,w)\in\mathbb{C}^n\times C^{1,\alpha}(\Delta)$ *small, there is a unique* $\epsilon\in C^{1,\alpha}(\Delta)$*, small, such that*

$$\partial_{\bar\tau}\epsilon-\lambda\partial_{\bar\tau}(\overline{w_o+w+\epsilon})=0, \tag{2.6}$$

that is

$$\epsilon(z)=\frac{1}{2\pi i}\iint_\Delta\frac{\lambda\partial_{\bar\tau}(\overline{w+\epsilon})}{\tau-z}d\tau d\bar\tau. \tag{2.7}$$

Moreover

$$||\epsilon||_{1,\alpha}=O(|w_o|+||w||_{1,\alpha}). \tag{2.8}$$

Proof. We rewrite (2.6) as $F=0$, recall that F is differentiable with $\partial_\epsilon F$ close to the identity, and apply the implicit function theorem in Banach spaces. □

Remark 2.4. We remark that if $\lambda\in C^\omega$, the space of real analytic functions, then, since F depends in a C^ω fashion on w_o, it follows that the solution $\epsilon=\epsilon_{w_o}$, $\mathbb{C}^n\to C^{1,\alpha}(\Delta)$ is C^ω. Also, for a fixed $w=w(\tau)\in C^\omega$, we see that $\epsilon_{w_o,w}$ is C^ω also with respect to τ since it solves the C^ω elliptic equation $\partial_{\bar\tau}\epsilon-\lambda\partial_{\bar\tau}(\overline{w+\epsilon})=0$.

3. Estimates of the deviation from linear discs

We consider an almost complex structure on X defined by a system of $T^{0,1}X$ vector fields $\bar L_j=\partial_{\bar z_j}+\sum_k\lambda_{kj}\partial_{z_k}$ with $(\lambda_{kj})|_0=0$.

We need to refine here the estimate (2.8) and specify the dependence of ϵ on the parameters w_o.

Proposition 3.1. *We have*

$$\partial_{w_o,\bar w_o}\epsilon=O(|w_o|,||w||_{1,\alpha}), \tag{3.1}$$
$$\partial_{w,\bar w}\epsilon=O(|w_o|,||w||_{1,\alpha}). \tag{3.2}$$

Proof. We have

$$\begin{aligned}|\partial_{w_o,\bar{w}_o}\epsilon(z)| &\leq \iint_\Delta \frac{|\lambda'||\partial_{\bar{\tau}}\bar{w}+\partial_{\bar{\tau}}\bar{\epsilon}|}{|\tau-z|}d\tau d\bar{\tau}\\ &\leq ||\lambda'||_0(||w||_{1,\alpha}+|w_o|)\\ &= O(||w||_{1,\alpha}+|w_o|),\end{aligned} \tag{3.3}$$

which yields at once (3.1).

As for (3.2) this follows from:

$$\begin{aligned}|\partial_{w,\bar{w}}\epsilon(z)| \leq \iint_\Delta \frac{|\lambda'||\partial_{\bar{\tau}}\bar{w}+\partial_{\bar{\tau}}\bar{\epsilon}|}{|\tau-z|}d\tau d\bar{\tau} + \iint_\Delta \frac{|\lambda|(1+|\partial_{\bar{\tau}}\bar{\epsilon}|)}{|\tau-z|}d\tau d\bar{\tau} \quad (3.4)\\ = O(|w_o|+||w||_{1,\alpha}).\end{aligned}$$

□

We will point our attention to the case of linear discs $w_o+w_1\tau$ where w_o and w_1 are vectors in TX. If we expand the mapping $(w_o,w_1)\to\epsilon_{w_o,w_1}$ with respect to w_o, w_1, we get

Corollary 3.2. *We have*

$$||\epsilon||_0 = O^2(|w_o|+|w_1|). \tag{3.5}$$

We fix now $w_1\in\mathbb{C}^n\simeq TX$, and let w_o vary in the plane $\mathbb{C}^{n-1}$ orthogonal to w_1. We denote by $A_{w_o}(\tau)$ the almost complex disc $w_o+w_1\tau+\epsilon(\tau)$ and also write A_{w_o} instead of the image set $A_{w_o}(\Delta)$.

Theorem 3.3. *The family of discs $\{A_{w_o}\}_{w_o\in\mathbb{C}^{n-1}}$ give a foliation of a neighborhood of* 0 *in X.*

Proof. We have to prove that for any point z of a neighborhood of 0 there are unique values of $w_o\in\mathbb{C}^{n-1}$ and $\tau\in\Delta$ such that $z=A_{w_o}(\tau)$.

We consider the mapping

$$\Delta\times\mathbb{C}^{n-1}\overset{G}{\to}X, \tag{3.6}$$

$$(\tau,w_o)\overset{G}{\to}w_o+w_1\tau+\epsilon(\tau). \tag{3.7}$$

We decompose $\mathbb{C}^n=\mathbb{C}\times\mathbb{C}^{n-1}$ with coordinates (w_1,w_o). The Jacobian G' of G takes the form

$$G'=\begin{pmatrix}w_1 & 0\\ 0 & \mathrm{id}\end{pmatrix}+\epsilon'$$

where

$$\epsilon'=\begin{pmatrix}\partial_{\tau,\bar{\tau}}\epsilon_1 & \partial_{w_o,\bar{w}_o}\epsilon_1\\ \partial_{\tau,\bar{\tau}}\epsilon_o & \partial_{w_o,\bar{w}_o}\epsilon_o.\end{pmatrix}$$

We use (2.8) to estimate the terms on the first column of ϵ' and (3.1) for those on the second. Thus ϵ' enters as an error term in G'; in particular G' is invertible (with $\det(G')\sim w_1$). □

Remark 3.4. In (2.8), (3.1), (3.2) we have in fact $O(||\lambda'||_0 O(|w_o|,||w||_{1,\alpha})$; in particular, if we suppose in addition to $(\lambda_{kj})|_0=0$, also $(\lambda'_{kj})|_0=0$, then we can replace "O" by "o" in Proposition 3.1 and Corollary 3.2.

Let $M = \partial\Omega$ be a hypersurface in $\mathbb{C}^n$ the boundary of a domain Ω which seats locally on one side of M. We assume that M is defined by $r = 0$ with $dr \neq 0$ and Ω by $r < 0$. We suppose that there exists at least one negative eigenvector, say w_1, of the Levi form $\mathcal{L}_M$ for the standard structure of $\mathbb{C}^n$, from the side of Ω, that is that $\mathcal{L}_M(w_1, \bar{w}_1)\lrcorner dr < 0$. We decompose $\mathbb{C}^n = \mathbb{C}_{w_1} \times \mathbb{C}^{n-1}_{w_o}$ and take complex coordinates in $\mathbb{C}^n$ with $w = (w_1, w_o)$, $w_o = (w', w_n)$ and $w = x + iy$ so that for the equation $r = 0$ of M we have for a suitable constant $c > 0$

$$r = y_n - (c|w_1|^2 + \dots) + o^2(w_1, w', x_n),$$

where the "dots" denote quadratic terms with respect to $\tilde{w}$. In particular

$$y_n \geq c|w_1|^2 - c'|w'|^2 + o^2(w_1, w', x_n) \text{ on } \partial\Omega.$$

Theorem 3.5. *Let M have at least one negative Levi eigenvalue for the standard structure of $X = \mathbb{C}^n$. Let us change the standard structure to a new J associated to a bundle $T^{0,1}X$ engendered by a system* (2.1) *of* $(0,1)$ *vector fields satisfying $\lambda_{ij}(0) = 0$. Assume that λ' is sufficiently small. Then for δ_1, $\delta_2 = \sqrt{c3c'}$ and $\delta_3 = \frac{c}{3}\delta_1^2$ sufficiently small, and for*

$$\begin{cases} |w_1| = \delta_1, \\ |y_n^o| \leq \delta_3, \\ |(x_n^o, w')| \leq \delta_2, \end{cases} \tag{3.8}$$

the family of J-holomorphic discs $\{A_{w_o}\}_{w_o} = \{w_o + w_1\tau + \epsilon(\tau)\}_{w_o}$, $\tau \in \Delta$, satisfy

$$\begin{cases} \cup_{w_o} A_{w_o} \text{ is a neighborhood of } 0, \\ \partial A_{w_o} \subset\subset \Omega. \end{cases} \tag{3.9}$$

Proof. The first of (3.9) is a consequence of Theorem 3.3. As for the second, we denote by π_{y_n} the projection along the y_n-axis. We have

$$\begin{cases} y_n|_{\partial\Omega \cap \pi_{y_n}^{-1}\pi_{y_n}(\partial A_{w_o})} \geq c\delta_1^2 - c'\delta_2^2 + o^2 > \frac{c}{2}\delta_1^2, \\ y_n|_{\partial A_{w_o}} \leq \frac{c}{3}\delta_1^2 + ||\lambda||_1 O^2. \end{cases} \tag{3.10}$$

If we then take λ such that $||\lambda||_1 < \frac{c}{2} - \frac{c}{3}$ we get $\partial A_{w_o} \subset\subset \Omega$. $\square$

Remark 3.6. It is clear that if $|\lambda'|$ is not small comparing with c, then since $\epsilon \sim |\lambda'||w_1|^2$, it follows that ∂A_{w_o} needs not to be contained in Ω.

4. Applications

We still consider an almost complex structure on X, its associated system of $T^{0,1}X$ vector fields $\bar{L}_j = \partial_{\bar{z}_j} + \sum_k \lambda_{kj}\partial_{z_k}$ with $(\lambda_{kj})|_0 = 0$, and the almost holomorphic functions on X that is the solutions of $\bar{L}f = 0\,\forall \bar{L} \in T^{0,1}X$. We will confine our attention to almost analytic discs which are perturbations of linear ones, namely

$A(\tau) = w_o + w_1\tau + \epsilon(\tau)$ where ϵ is a solution of (2.3). We observe that $A_*\partial_{\bar\tau} \in T^{0,1}X$ and therefore, if f is almost holomorphic

$$\partial_{\bar\tau}(f \circ A) = (A_*\partial_{\bar\tau})f|_A = 0; \tag{4.1}$$

thus, $f \circ A$ is indeed holomorphic. An easy consequence of this property is the content of the following

Proposition 4.1. *The almost holomorphic functions satisfy the analytic continuation principle, that is if their domain is connected and if they vanish on an open subset, they are identically* 0*.*

Proof. Let $\Omega' \subset \Omega$ be open sets, Ω connected, and let f be an almost holomorphic function on Ω such that $f|_{\Omega'} \equiv 0$. Ω is supposed to be small so that $|\lambda|$ and $|\lambda'|$ are small and, in particular, the conclusions of Theorem 2.3 hold. (Otherwise, the proof can be carried on by an argument of partition of the unity.) Let z^o be a point in Ω' and let z^1 be any other point in Ω. Let Γ be a real curve connecting z^o to z^1, and let ξ be the "extremal" point on Γ to which 0 propagates. Let $w_1 := T_\xi\Gamma$ and let A_ξ be the almost complex disc through ξ in direction w_1 that is $A_\xi = w_o + w_1\tau + \epsilon(\tau)$. Remember that

$$\begin{cases} f \circ A_\xi \text{ is holomorphic,} \\ f \circ A_\xi \equiv 0 \text{ in some open part of } \Delta. \end{cases} \tag{4.2}$$

Since $f \circ A_\xi$ is holomorphic, then $f \circ A_\xi \equiv 0$ which contradicts the maximality of ξ unless ξ is z^1 itself. $\square$

Let M be a hypersurface in $X = \mathbb{C}^n$ the boundary $M = \partial\Omega$ of a domain Ω which seats locally on one side of M. We assume that M is defined by $r = 0$ with $dr \neq 0$ and Ω by $r < 0$.

Theorem 4.2. *Assume that M has at least one negative Levi eigenvalue from the side Ω for the standard structure of $\mathbb{C}^n$. Let J be a almost complex structure on X associated to a bundle $T^{0,1}X$ engendered by a system* (2.1) *of* $(0,1)$ *vector fields with $\lambda(0) = 0$ and with λ real analytic. We assume that $||\lambda||_1$ is so small that the conclusions of Theorem* 3.5 *hold. Then, for any almost holomorphic f on Ω in a neighborhood of* 0*, there exists $\tilde f$ defined over a* (*possibly smaller*) *neighborhood of* 0 *in X, such that*

$$\begin{cases} \tilde f|_{A_{w_o}} \textit{ is holomorphic for any } w_o \in \mathbb{C}^{n-1}, \\ \tilde f|_\Omega \equiv f. \end{cases}$$

Proof. According to Theorem 3.5, if $|\lambda'|$ is small and if $|w_1| = \delta_1,\ |y_n^o| \le \delta_3$ and $|(x_n^o, w')| \le \delta_2$ for suitable δ's, the discs A_{w_o} fill a neighborhood of 0 and satisfy

$$\partial A_{w_o} \subset\subset \Omega.$$

After this preparation we are ready to carry out the proof of our theorem. We define the extension $\tilde{f}$ through the Cauchy integral

$$\tilde{f}(A_{w_o}(\zeta)) = \frac{1}{2\pi i}\int_{\partial\Delta}\frac{f\circ A_{w_o}(\tau)}{\tau-\zeta}d\tau. \tag{4.3}$$

We observe that

- The integral (4.3) makes sense since each ∂A_{w_o} is contained in Ω.
- The collection of the functions $\tilde{f}\circ A_{w_o}$ gives rise to a well-defined function $\tilde{f}$ on a neighborhood of 0 because the family of discs A_{w_o} is a foliation.

We also remark that

$$\tilde{f}|_{A_{w_o}} = f|_{A_{w_o}} \text{ when } A_{w_o}\subset\Omega, \tag{4.4}$$

due to the Cauchy formula for the holomorphic function $f\circ A_{w_o}$. But then, since $\tilde{f}|_{A_{w_o}} - f|_{A_{w_o}}$ depends in a C^ω fashion on w_o, and since it is 0 for all w_o on an open set, then it is indeed identically 0 on a neighborhood of 0 by analytic continuation. The proof is complete. □

Remark 4.3. If $\lambda(0) = 0$ but λ' is not small enough, the construction in Theorem 4.2 might break. For instance, in $\mathbb{C}^2$, let us consider the transformation

$$\begin{cases} z_1 \to z_1 + k\bar{z}_1\bar{z}_2, \\ z_2 \to z_2, \end{cases}$$

where k is a big constant. Under this transformation, the antiholomorphic derivatives transform according to the rule

$$\begin{cases} \partial_{\bar{z}_1} \to \partial_{\bar{z}_1} + k\bar{z}_2\partial_{z_1}, \\ \partial_{\bar{z}_2} \to \partial_{\bar{z}_2} + k\bar{z}_1\partial_{z_1}. \end{cases}$$

Thus the corresponding matrix λ can be described by

$$\lambda = k\begin{pmatrix} \bar{z}_2 & 0 \\ \bar{z}_1 & 0 \end{pmatrix}$$

In particular, $\lambda|_0 = 0$ and $\partial_z\lambda|_0 = 0$ and hence the Levi form is preserved. Consider in this setting the domain Ω defined by $\mathsf{Im}\, z_2 < |z_1|^2$, whose boundary satisfies $\mathcal{L}_{\partial\Omega}(w_1, \bar{w}_1) < 0$. According to the proof of Theorem 4.2, we try to extend almost holomorphic functions on Ω by means of discs of the form

$$A_{w_o}(\tau) = w_o + w_1\tau + \epsilon(\tau),\ \tau\in\Delta,$$

for w_1 fixed and small in $\mathbb{C}\times\{0\}$, and for all $w_o\in\{0\}\times\mathbb{C}$ satisfying $|w_o|\le|w_1|^2$. Now, for the error terms $\epsilon = (\epsilon_1, \epsilon_2)$ we have the equations

$$\epsilon_1 = k\iint_\Delta \frac{\overline{w_o+\epsilon_2}(\partial_{\bar\zeta}\overline{\epsilon_1+w_1})}{\zeta-\tau}d\zeta d\bar\zeta, \quad \epsilon_2 = k\iint_\Delta \frac{\overline{w_1\tau+\epsilon_1}(\partial_{\bar\zeta}\overline{\epsilon_1+w_1})}{\zeta-\tau}d\zeta d\bar\zeta.$$

In conclusion $|\epsilon_2| \sim k|w_1|^2$ and hence, we do not have the condition $\partial A \subset\subset \Omega$ satisfied.

Remark 4.4. The hypothesis $\lambda \in C^\omega$ cannot be removed in the statement of Theorem 4.2. For example, let $\chi : \mathbb{R} \to \mathbb{R}$ be a smooth function such that χ vanishes identically if $t \leq 0$ and suppose further that $\chi'(t) \neq 0$ for $t \geq 0$. Let M be the hypersurface in $\mathbb{C}^2$ of equation $\mathsf{Im}\, z_2 = |z_1|^2$. We consider the modified structure in $\mathbb{C}^2$ whose system of $(0,1)$ vector fields is spanned by the basis

$$\left\{\bar{\partial}_{z_1}, \bar{\partial}_{z_2} + \chi(\mathsf{Im}\, z_2 - |z_1|^2)\partial_{z_1}\right\} \tag{4.5}$$

It is clear that the new structure coincides with the standard one over the domain Ω defined by $\mathsf{Im}\, z_2 - |z_1|^2 < 0$; in other words we have $\lambda|_\Omega \equiv 0$. If we consider the Lie-Span $\mathcal{V}$ of the above system of $(0,1)$ vector fields, we see that

$$\mathcal{V} = \begin{cases} \langle \bar{\partial}_{z_1}, \bar{\partial}_{z_2} \rangle \text{ if } y_2 - |z_1|^2 \leq 0 \\ \langle \bar{\partial}_{z_1}, \bar{\partial}_{z_2}, \partial_{z_1} \rangle \text{ if } |z_1|^2 - y_2 < 0 \end{cases}$$

In particular, almost complex functions coincide with usual complex functions over Ω and with complex functions which are constant with respect to z_1 over $\mathbb{C}^2 \setminus \Omega$. Also, if we consider the family of linear discs $A_{w_o}(\tau) = w_o + w_1\tau + \epsilon(\tau)$, they are already almost complex (or, in other words, we have $\epsilon = 0$ in this case); in particular, $\mathcal{V}|_{A_{w_o}} \supset \mathbb{C} \otimes TA_{w_o}$. Thus, almost complex functions along these discs are constant. It follows that not all almost complex functions extend from Ω to the discs A_{w_o}.

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Hyperbolic Problems and Regularity Questions
Trends in Mathematics, 39–47

Vorticity and Regularity for Solutions of Initial-boundary Value Problems for the Navier–Stokes Equations

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Abstract. In reference [7], among other side results, we prove that the solution of the evolution Navier–Stokes equations (1.1) under the Navier (or slip) boundary condition (1.2) is necessarily regular if the direction of the vorticity is 1/2-Hölder continuous with respect to the space variables. In this notes we show the main steps in the proof and made some comments on the above problem under the non-slip boundary condition (3.2).

1. Introduction

In the sequel we denote by $\omega(x,t) = \nabla \times u(x,t)$ the vorticity of the velocity field u and define the direction of the vorticity ξ as

$$\xi(x) = \frac{\omega(x)}{|\omega(x)|}\,.$$

In general we will use the notation

$$\widehat{z} = \frac{z}{|z|}$$

if $|z| \neq 0$. Hence $\xi = \widehat{\omega}$. Moreover, denote by $\theta(x,y,t)$ the angle between the vorticity ω at two distinct points x and y at time t.

In reference [11] Constantin and Fefferman prove that weak solutions to the evolution Navier–Stokes equations in the whole of $\mathbf{R}^3$ are regular if the direction of the vorticity is Lipschitz continuous with respect to the space variables:

$$\sin\theta(x,y,t) \leq c|x-y|\,.$$

In reference [6], among other side results, the authors prove that 1/2-Hölder continuity

$$\sin\theta(x,y,t) \leq c|x-y|^{1/2}$$

is sufficient to guarantee the regularity of weak solutions. Main ingredients in the proof of the above result are Biot-Savart Law and a meaningful formula introduced in reference [10]. See equation (7) in [11]. We remark that in reference [7] we bypass the use of this formula by estimating directly the integral on the right-hand side of equation (2.2).

A main open problem remains of the possibility of extending the same kind of results to boundary value problems. In reference [7] we succeed in making this extension to the well-known Navier (or slip) boundary condition. More precisely, the following result is proved. Notation is given below.

Theorem 1.1. *Let $u_0 \in V$ and let u be a weak solution of the Navier–Stokes equations in $[0,\, T) \times\, \mathbf{R}^3_+$, namely,*

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nu \Delta u + \nabla p = 0 \quad in\ \mathbf{R}^3_+ \times [0, T), \\ \nabla \cdot\, u = 0 \quad in\ \mathbf{R}^3_+ \times [0, T), \\ u(x, 0) = u_0(x) \quad in\ \mathbf{R}^3_+, \end{cases} \tag{1.1}$$

endowed with the slip boundary condition

$$\begin{cases} \quad u_3 = 0, \\ \quad \nu \frac{\partial u_j}{\partial x_3} = 0, \quad 1 \leq j \leq 2. \end{cases} \tag{1.2}$$

Let $\beta \in [0, 1/2]$ and assume that

$$|\sin \theta(x, y, t)| \leq\, c|x - y|^\beta \tag{1.3}$$

in the region where the vorticity at both points x and y is larger than an arbitrary fixed positive constant K. Moreover, suppose that

$$\omega \in L^2(0, T; L^r), \tag{1.4}$$

where

$$r = \frac{3}{\beta + 1}. \tag{1.5}$$

Then the solution u is strong in $[0, T]$ and, consequently, is regular. If $\beta =\, 1/2$ the assumption (1.4) is superfluous.

The last claim follows from the fact that weak solutions satisfy (1.4) for $r =\, 2$.

Next we recall some definitions concerning the *slip boundary condition.* For more details see [3, 4, 17]. We remark that the standard functional framework in studying the boundary condition (1.2) is

$$V =\, \left\{ v \in [H^1(\mathbf{R}^3_+)]^2 \times H^1_0(\mathbf{R}^3_+) : \nabla \cdot v =\, 0 \right\}\, .$$

See [3].

Even though we consider here the Navier–Stokes equations in the half-space $\mathbf{R}^3_+ \,=\, \left\{ x \in \mathbf{R}^3 \,:\, x_3 \,>\, 0 \right\}$ it is suitable to describe the slip boundary condition

(1.7) in the general case of an open set Ω in $\mathbf{R}^3$. Γ denotes the boundary of Ω and $\underline{n}$ the unit external normal to Γ. We denote by

$$T = -p\,I + \nu(\nabla u + \nabla u^T)$$

the stress tensor, by $\underline{t} = T \cdot \underline{n}$ the stress vector and define the linear operator

$$\underline{\tau}(u) = \underline{t} - (\underline{t} \cdot \underline{n})\underline{n}. \tag{1.6}$$

The vector field $\underline{\tau}(u)$ is tangential to the boundary and independent of the pressure p.

The slip boundary condition reads

$$\begin{cases} (u \cdot \underline{n})_{|\Gamma} = 0, \\ \underline{\tau}(u)_{|\Gamma} = 0\,. \end{cases} \tag{1.7}$$

When $\Omega = \mathbf{R}^3_+$, the equations (1.7) have the form (1.2). See [3], Equation (2.2).

The literature related to the slip boundary condition (1.7) is wide. This boundary condition is an appropriate model for many important flow problems. Besides the pioneering mathematical contribution [17] by Solonnikov and Ščadilov, this boundary condition has been considered by many authors. See [3, 4, 5] and references therein as, for instance, [1, 9, 12, 13, 14, 15, 16, 18].

It is worth noting that our proof may be adapted to other boundary conditions. However, the extension to the non-slip boundary condition (3.2) under the *sole* assumption $\sin\theta(x, y, t) \leq c|x - y|^{1/2}$ remains an open problem, even if some partial results are available. See the last section.

2. Sketch of the proof of Theorem 1.1

We denote by $|\cdot|_p$ the canonical norm in the Lebesgue space $L^p := L^p(\mathbf{R}^3)$, $1 \leq p \leq \infty$. $H^s := H^s(\mathbf{R}^3)$, $0 \leq s$, denotes the classical Sobolev spaces. Scalar and vector function spaces are indicated by the same symbol.

From now on we set

$$\Omega = \mathbf{R}^3_+ \quad \text{and} \quad \Gamma = \left\{ x \in \mathbf{R}^3 : x_3 = 0 \right\}.$$

For convenience, we mostly will use the Ω, Γ notation.

Since $u_0 \in H^1$, the solution is strong, hence regular, in $[0, \tau)$, for some $\tau > 0$. Let $\tau \leq T$ be the maximum of these values. We show that, under this hypothesis, u is strong in $[0, \tau]$. Hence, by a continuation principle, u is strong in $[\tau, \tau + \varepsilon)$. This shows that $\tau = T$. Without loss of generality we assume that the solution u is regular in $[0, T)$ and we prove that this implies regularity in $[0, T]$.

By taking the curl of both sides of the first equation (1.1) we find, for each $t < T$,

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\,\omega - \nu\Delta\omega = (\omega \cdot \nabla)\,u \tag{2.1}$$

in $\mathbf{R}^3_+$. Moreover, by taking the scalar product in L^2 of both sides of (2.1) with ω, we get

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 = \int_\Omega (\omega\cdot\nabla)\, u\, \cdot \omega(x)\, dx. \tag{2.2}$$

Note that

$$-\nu\int_\Omega \Delta\omega\cdot\omega\, dx = \nu|\nabla\omega|_2^2 + \nu\int_\Gamma \frac{\partial\,\omega}{\partial\, x_3}\cdot\omega\, d\Gamma \tag{2.3}$$

since $\underline{n} = (0,0,-1)$. Under the boundary condition (1.2) it readily follows that

$$\int_\Gamma \frac{\partial\,\omega}{\partial\, x_3}\cdot\omega\, d\Gamma = \int_\Gamma \frac{\partial\,\omega_3}{\partial\, x_3}\cdot\omega_3\, d\Gamma = 0\,.$$

However, under the non-slip boundary condition (3.2) one gets

$$\nu\int_\Gamma \frac{\partial\,\omega}{\partial\, x_3}\cdot\omega\, d\Gamma = \frac{\nu}{2}\,\frac{d}{d\,x_3}\int_{\Gamma(x_3)} (\omega_1^2 + \omega_2^2)\, d\Gamma\,. \tag{2.4}$$

If we are able to control this quantity in a suitable way, then Theorem 1.1 applies to the non-slip boundary condition as well, as easily shown by a simple adaptation of the proofs given here. See [8].

Set, for each triad (j,k,l), $j,k,l\in\{1,2,3\}$,

$$\epsilon_{ijk} = \begin{cases} \;\;1 \text{ if } (i,j,k) \text{ is an even permutation}\,, \\ -1 \text{ if } (i,j,k) \text{ is an odd permutation}\,, \\ \;\;0 \text{ if two indexes are equal}\,. \end{cases} \tag{2.5}$$

One has

$$(a\times b)_j = \epsilon_{jkl}\, a_k\, b_l \tag{2.6}$$

and

$$(\nabla\times v)_j = \epsilon_{jkl}\,\frac{\partial\, v_l}{\partial\, x_k}\,, \tag{2.7}$$

where here, and in the sequel, the usual convention about summation of repeated indices is assumed.

Since

$$-\,\Delta\, u = \nabla\times\,(\nabla\times u) - \nabla\,(\nabla\cdot\, u)\,, \tag{2.8}$$

it follows that

$$\begin{cases} -\Delta\, u = \nabla\times\omega \quad \text{in} \quad \Omega\,; \\ \frac{\partial u_1}{\partial x_3} = \frac{\partial u_2}{\partial x_3} = 0 \quad \text{in} \quad \Gamma\,, \\ u_3 = 0 \quad \text{in} \quad \Gamma\,, \end{cases} \tag{2.9}$$

for each t.

In the sequel

$$G(x,y) = \frac{1}{4\,\pi}\left(\frac{1}{|x-y|} - \frac{1}{|x-\overline{y}|}\right) \tag{2.10}$$

denotes the Green's function for the Dirichlet boundary value problem in the half-space, where

$$\overline{y} = (y_1,\, y_2,\, -y_3)\,,$$

and

$$N(x,y) = \frac{1}{4\,\pi}\left(\frac{1}{|x-\,y|} + \frac{1}{|x-\,\overline{y}|}\right) \tag{2.11}$$

denotes the classical Neumann's function for the half-space $\mathbf{R}^3_+$.

For $j = 1, 2, 3$ we set

$$\begin{cases} a_j(x) = -\frac{1}{4\,\pi}\int_\Omega \epsilon_{jkl}\,\frac{x_k-\,y_k}{|x-\,y|^3}\,\omega_l(y)\,dy\,, \\ b_j(x) = \frac{1}{4\,\pi}\int_\Omega \epsilon_{jkl}\,\epsilon_k\,\frac{x_k-\,\overline{y}_k}{|x-\,\overline{y}|^3}\,\omega_l(y)\,dy\,, \end{cases} \tag{2.12}$$

where

$$\epsilon_1 = \epsilon_2 = 1\,,\, \epsilon_3 = -1\,.$$

In our notation we often drop the symbol t specially when it may be viewed as a parameter. One has the following result:

Lemma 2.1. *For each $x \in \Omega$*

$$\begin{aligned} &\frac{\partial\,a_j(x)}{\partial\,x_i}\,\omega_i(x)\,\omega_j(x) \\ &= \frac{3}{4\,\pi}\,P.V.\int_\Omega \widehat{(x-\,y)}\cdot\,\omega(x)\;\mathrm{Det}\left(\widehat{(x-\,y)},\omega(y),\omega(x)\right)\,\frac{dy}{|x-\,y|^3} \end{aligned} \tag{2.13}$$

and

$$\begin{aligned} &\frac{\partial\,b_j(x)}{\partial\,x_i}\,\omega_i(x)\,\omega_j(x) \\ &= \frac{1}{4\,\pi}\int_\Omega \mathrm{Det}\,(\omega(x),\overline{\omega}(x),\omega(y))\,\frac{dy}{|x-\,\overline{y}|^3} \\ &\quad - \frac{3}{4\,\pi}\int_\Omega \widehat{(x-\,\overline{y})}\cdot\,\omega(x)\;\mathrm{Det}\left(\widehat{(x-\,\overline{y})},\omega(y),\omega(x)\right)\,\frac{dy}{|x-\,\overline{y}|^3}\,. \end{aligned} \tag{2.14}$$

For the proof see [7].

For $j = 3$ it follows from (2.9) that

$$u_j(x) = \int_\Omega G(x,\,y)\,(\nabla\times\omega(y))_j\,dy\,, \tag{2.15}$$

By appealing to (2.7) and by taking into account that $G(x,\,y) = 0$ if $y\in\Gamma$, an integration by parts yields

$$u_j(x) = -\int_\Omega \epsilon_{jkl}\,\frac{\partial\,G(x,y)}{\partial\,y_k}\,\omega_l(y)\,dy\,. \tag{2.16}$$

Hence, for $j = 3$, (2.10) shows that

$$u_j(x) = a_j(x) + b_j(x)\,. \tag{2.17}$$

By appealing to (2.7) and (2.9) it follows that

$$\begin{cases} -\Delta\,u_j = \epsilon_{jkl}\,\frac{\partial\,\omega_l}{\partial\,x_k}\,, & \text{in} \quad \Omega\,, \\ \frac{\partial u_j}{\partial x_3} = 0, & \text{in} \quad \Gamma\,, \end{cases} \tag{2.18}$$

where $j = 1$ or $j = 2$. From (2.18) one gets

$$u_j(x) = \int_\Omega N(x,y)\,\epsilon_{jkl}\,\frac{\partial\,\omega_l(y)}{\partial\,y_k}\,dy\,, \tag{2.19}$$

for $j = 1,\,2$. Hence, for $j = 1,2$,

$$u_j(x) = -\int_\Omega \epsilon_{jkl}\,\frac{\partial\,N(x,y)}{\partial\,y_k}\,\omega_l(y)\,dy\,+\,\gamma_j(x)\,, \tag{2.20}$$

where

$$\gamma_j(x) = \int_\Gamma N(x,y)\,\epsilon_{jkl}\,\omega_l(y)\,n_k\,dy \tag{2.21}$$

is defined for $j = 1,2,3$ and $n_k = (0,0,-1)$. Note that $\gamma_3(x) = 0$.

It readily follows, by appealing to (2.17) when $j = 3$, that

$$u_j(x) = a_j(x) - \epsilon_j\,b_j(x) + \gamma_j(x)\,, \quad j = 1,2,3. \tag{2.22}$$

From (2.22), (2.13) and (2.14) straightforward calculations show that

$$\begin{aligned}
&((\omega\cdot\nabla)\,u\cdot\omega)\,(x) \equiv \tfrac{\partial\,u_j(x)}{\partial\,x_i}\,\omega_i(x)\omega_j(x)\\
&= -\tfrac{3}{4\,\pi}\,P.V.\int_\Omega\Big((\widehat{x-y})\cdot\omega(x)\Big)\;\mathrm{Det}\,\Big((\widehat{x-y}),\omega(x),\omega(y)\Big)\;\tfrac{dy}{|x-\,y|^3}\\
&\quad-\tfrac{3}{4\,\pi}\int_\Omega\Big((\widehat{x-\overline{y}})\cdot\omega(x)\Big)\;\mathrm{Det}\,\Big((\widehat{\overline{x-\overline{y}}}),\,\overline{\omega}(x),\,\omega(y)\Big)\;\tfrac{dy}{|x-\,\overline{y}|^3}\\
&\quad+\tfrac{\partial\,\gamma_j(x)}{\partial\,x_i}\,\omega_i(x)\omega_j(x)\,.
\end{aligned} \tag{2.23}$$

A careful argument, see [7], shows that

$$\begin{aligned}
&\tfrac{\partial\,\gamma_j(x)}{\partial\,x_i}\,\omega_i(x)\omega_j(x)\\
&= \tfrac{1}{2\,\pi}\,P.V.\int_\Gamma(\widehat{x-y})\cdot\omega(x)\;\;\mathrm{Det}\,(n(y),\omega(x),\omega(y))\;\tfrac{dy}{|x-\,y|^2}\,.
\end{aligned} \tag{2.24}$$

Since, for $y \in \Gamma$, $n(y)$ and $\omega(y)$ are parallel, the last term in the right-hand side of (2.23) vanishes. Hence we get the following result.

Lemma 2.2. *Under the above hypothesis one has the following identity.*

$$\begin{aligned}
&((\omega\cdot\nabla)\,u\cdot\omega)\,(x)\\
&= -\tfrac{3}{4\,\pi}\,P.V.\int_\Omega(\widehat{x-y})\cdot\omega(x)\;\;\mathrm{Det}\,\Big((\widehat{x-y}),\omega(x),\omega(y)\Big)\;\tfrac{dy}{|x-\,y|^3}\\
&\quad-\tfrac{3}{4\,\pi}\int_\Omega\Big((\widehat{x-\overline{y}})\cdot\omega(x)\Big)\;\mathrm{Det}\,\Big((\widehat{\overline{x-\overline{y}}}),\,\overline{\omega}(x),\,\omega(y)\Big)\;\tfrac{dy}{|x-\,\overline{y}|^3}\\
&=: I_1(x) + I_2(x)\,.
\end{aligned} \tag{2.25}$$

In the following two lemmas the integrals over Ω of the above quantities I_1 and I_2 are estimated. For the proofs see [7].

Lemma 2.3. *For each $t \in (0,\,T)$ the following estimate holds.*

$$\left|\int_\Omega I_1(x)\,dx\right| \le \frac{\nu}{4}\,|\nabla\,\omega|_2^2 + c\,\Big(K + \nu^{-\frac{3}{5}}\,K^{\frac{4}{5}}|\omega|_2^{\frac{4}{5}} + \nu^{-1}\,|\omega|_r^2\Big)\;|\omega|_2^2\,. \tag{2.26}$$

The proof of this lemma follows some known arguments used in references [11] and [6]. On the contrary, the proof of the next lemma appeals to new ideas, and is strongly related to the boundary conditions.

Lemma 2.4. *For each $t \in (0, T)$ the following estimate holds:*

$$\left| \int_\Omega I_2(x)\, dx \right| \leq \frac{\nu}{4} |\nabla \omega|_2^2 + c \left(K + \nu^{-\frac{3}{5}} K^{\frac{4}{5}} |\omega|_2^{\frac{4}{5}} + \nu^{-1} |\omega|_r^2 \right) |\omega|_2^2 . \tag{2.27}$$

Finally, from (2.2), (2.25) and from Lemmas 2.3 and 2.4 it readily follows that

$$\frac{1}{2}\frac{d}{dt}|\omega|_2^2 + \frac{\nu}{2} |\nabla\omega|_2^2 \leq c \left(K + \nu^{-\frac{3}{5}} K^{\frac{4}{5}} |\omega|_2^{\frac{4}{5}} + \nu^{-1} |\omega|_r^2 \right) |\omega|_2^2 . \tag{2.28}$$

Since $|\omega|_2^{\frac{4}{5}}$ and $|\omega|_r^2$ are integrable in $(0, T)$ a well-known argument shows that

$$u \in L^\infty(0,T;H^1) \cap L^2(0,T;H^2) .$$

3. An open problem: The slip boundary condition

The method introduced in reference [7] to obtain suitable estimates for the right-hand side of equation (2.2) is not particularly lied to the slip boundary condition, and may be used to treat other boundary conditions as well. The extension from the slip boundary condition to the non-slip boundary condition of the crucial estimates concerning the “non-linear” term $(\omega \cdot \nabla)\, u \cdot \omega$ is simpler in the last case than in the former one. This extension is done in reference [8]. In this last reference we also succeed in replacing the half-space by a regular open set Ω. This is done by appealing to the structure of the Green function for the Poisson equation under the Dirichlet boundary condition. Nevertheless, we are not able to extend to the non slip boundary condition the 1/2-Hölder sufficient condition for regularity without an additional assumption, see Theorem 3.1 below. This point should be considered in a deeper form, possibly by tacking into account suitable physical arguments. The new obstacle here is not due to the “non-linear” term $(\omega \cdot \nabla)\, u \cdot \omega$ or to the presence of a non-flat boundary, but to the ”additional” boundary integral that appears on the left-hand side of equation (2.2). In fact, under the non-slip boundary condition the equation (2.2) should be replaced by

$$\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 - \nu \int_\Gamma \frac{\partial\omega}{\partial n} \cdot \omega \, d\Gamma = \int_\Omega (\omega \cdot \nabla)\, u \cdot \omega \, dx. \tag{3.1}$$

The boundary integral in equation (3.1) is due to the combination of viscosity with adherence to the boundary.

In reference [8] we prove the following result:

Theorem 3.1. *Let Ω be a bounded, connected, open set in $\mathbf{R}^3$, locally situated on one side of its boundary Γ, a manifold of class $C^{2,\alpha}$ and $u_0 \in H_0^1(\Omega)$ satisfy $\nabla \cdot u_0 = 0$.*

Let u be a weak solution of the Navier–Stokes equations in $[0,\,T)\times\Omega$ under the non-slip boundary condition

$$u = 0 \quad on \quad \Gamma\,. \tag{3.2}$$

Let $\beta\in[0,1/2]$ and assume that (1.3) holds in the region where the vorticity at both points x and $x+y$ is larger than an arbitrary fixed positive constant K. Moreover, suppose that (1.4) is satisfied, where r is given by (1.5). Then the following estimate holds:

$$\frac{d}{dt}|\omega|_2^2 + \nu|\nabla\omega|_2^2 - \frac{\nu}{2}\int\limits_{\Gamma}\frac{\partial|\omega|^2}{\partial n}\,d\Gamma \leq c\,h(t)\,|\omega|_2^2\,. \tag{3.3}$$

If, in addition, an upper bound of the form

$$\int\limits_{\Gamma}\frac{\partial|\omega|^2}{\partial n}\,d\Gamma \leq 2\int\limits_{\Omega}|\nabla\,\omega|^2\,dx + B(t)\int\limits_{\Omega}|\omega|^2\,dx \tag{3.4}$$

holds for some $B(t)\in L^1(0,\,T)$, then u is necessarily regular in $]0,\,T]$.

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Hyperbolic Problems and Regularity Questions
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Exponential Decay and Regularity for SG-elliptic Operators with Polynomial Coefficients

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Abstract. We study the exponential decay and the regularity for solutions of elliptic partial differential equations $Pu = f$, globally defined in $\mathbb{R}^n$. In particular, we consider linear operators with polynomial coefficients which are SG-elliptic at infinity. Starting from f in the so-called Gelfand-Shilov spaces, the solutions $u \in \mathcal{S}$ of the equation are proved to belong to the same classes. Proofs are based on a priori estimates and arguments on the Newton polyhedron associated to the operator P.

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1. Introduction

The aim of this paper is to investigate the regularity and the decay at infinity for the solutions to some partial differential equations $Pu = f$ globally defined in $\mathbb{R}^n$. Namely, assuming f to have a prescribed regularity and behavior at infinity, we ask whether u has the same properties. Concerning the behavior at infinity, we are interested in exponential decay for the solutions to $Pu = f$. The main interest for this problem comes hystorically from Quantum Mechanics, where the exponential decay of eigenfunctions of Schrödinger operators has been intensively studied, see for instance Agmon [1], Hislop and Sigal [11] and the references quoted therein. Starting from this point, many authors obtained results in the same direction for more general classes of elliptic operators, see Helffer and Parisse [10], Martinez [13], Rabier [17], Rabier and Stuart [18]. We also mention results by Bona and Li [4] on exponential decay and uniform analyticity of travelling waves, solving semilinear elliptic equations, (cf. also [3, 9]). Let us observe that the estimates

$$|f(x)| \leq Ce^{-\varepsilon|x|}, \qquad x \in \mathbb{R}^n \tag{1.1}$$

for some positive constants C, ε, are equivalent to $|x^\alpha f(x)| \leq C\varepsilon^{-|\alpha|}\alpha!, x \in \mathbb{R}^n, \alpha \in \mathbb{Z}_+^n$ for a new constant $C > 0$ independent of α. This suggests to combine (1.1) with uniform analyticity estimates. To this end, a natural and general functional frame is given by the spaces of Gelfand–Shilov type (cf. the classical book of Gelfand and Shilov [8], see also Mitjagin [14], Avantaggiati [2], Pilipovic [16], where functional properties of $S_\mu^\nu(\mathbb{R}^n)$ and of the dual spaces $S_\mu^{\nu\prime}(\mathbb{R}^n)$ of *tempered ultradistributions* have been investigated). We recall that $f \in S_\mu^\nu(\mathbb{R}^n)$ iff $f \in C^\infty(\mathbb{R}^n)$ and one can find $C > 0$ such that

$$\sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta f(x)| \leq C^{|\alpha|+|\beta|+1}(\alpha!)^\mu(\beta!)^\nu, \qquad \alpha, \beta \in \mathbb{Z}_+^n, \tag{1.2}$$

or equivalently, there exist $C > 0$, $\varepsilon > 0$ such that

$$\left|\partial_x^\beta f(x)\right| \leq C^{|\beta|+1}(\beta!)^\nu e^{-\varepsilon|x|^{1/\mu}} \tag{1.3}$$

for all $x \in \mathbb{R}^n, \beta \in \mathbb{Z}_+^n$. Recently (cf. [5]), global Gevrey regularity and subexponential decay in the space $S_\mu^\mu(\mathbb{R}^n), \mu > 1$ have been proved for SG-elliptic pseudodifferential operators, both by means of the construction of parametrices for P and by introducing a suitable wave front set for tempered ultradistributions of $S_\mu^{\mu\prime}(\mathbb{R}^n)$. The techniques in [5] are not applicable to the limiting case $\mu = 1$ in view of the lack of global calculus for analytic pseudodifferential operators. On the other hand, in [3, 9], the exponential decay and uniform analytic regularity estimates for solutions to some classes of semilinear partial differential equations imply $u \in S_1^1(\mathbb{R}^n)$ provided one assumes a priori that $\langle x\rangle^\tau u$ belongs to $H^s(\mathbb{R}^n)$ with $s > n/2$, for some positive τ. The main aim of the present paper is to derive $S_1^1(\mathbb{R}^n)$ regularity of the solutions of linear SG-elliptic partial differential operators with polynomial coefficients provided the right-hand side f in a subclass of $S_1^1(\mathbb{R}^n)$. We propose a rather simple proof, which uses basic a priori estimates for SG-elliptic differential operators and a suitable use of commutators, based in particular on the form of the Newton polyhedra associated to SG operators. An example demonstrating the sharpness of our estimates is also given. Finally, by similar methods, we derive $S_1^1(\mathbb{R}^n)$ regularity of solutions to polynomial nonlinear perturbations of the model SG elliptic operator $(1 + |x|^2)(1 - \Delta)$.

2. Main results

Consider in $\mathbb{R}^n$ the partial differential operator with polynomial coefficients

$$P = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m}} c_{\alpha\beta} x^\alpha D^\beta, \tag{2.1}$$

where $m > 0$ is a fixed positive integer. The operator P can be studied in the frame of the so-called SG-calculus, see Parenti [15], Schrohe [19], Cordes [6], Egorov and Schulze [7], Cappiello and Rodino [5]; let us recall some basic facts.

We assume that P is SG globally elliptic, namely, there exist $C > 0$ and $R > 0$ such that

$$\left| \sum_{\substack{|\alpha| \le m \\ |\beta| \le m}} c_{\alpha\beta} x^\alpha \xi^\beta \right| \ge C \langle x \rangle^m \langle \xi \rangle^m, \qquad |x| + |\xi| \ge R. \tag{2.2}$$

where $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}, \langle \xi \rangle = (1 + |\xi|^2)^{\frac{1}{2}}$. Global ellipticity in the previous sense implies both local regularity and asymptotic decay of the solutions, namely we have the following basic result: $Pu = f \in \mathcal{S}(\mathbb{R}^n)$ for $u \in \mathcal{S}'(\mathbb{R}^n)$ implies actually $u \in \mathcal{S}(\mathbb{R}^n)$. More precise statements are possible in terms of a suitable scale of weighted Sobolev spaces. We shall limit ourselves here to recall from [6, 7, 15, 19] the following a priori estimate for L^2-norms, which we shall use later: if P is globally elliptic of order m, then there exists a positive constant $C^\star$ such that for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\sum_{\substack{|\alpha| \le m \\ |\beta| \le m}} \|x^\alpha D^\beta u\| \le C^\star \left(\|Pu\| + \|u\| \right). \tag{2.3}$$

We introduce a subclass of $S_1^1(\mathbb{R}^n)$.

Definition 2.1. We denote by $S_1^{1\star}(\mathbb{R}^n)$ the class of all functions $f \in \mathcal{S}(\mathbb{R}^n)$ such that, for every $N \in \mathbb{Z}_+$ and for some constant $C > 0$ independent of N

$$\|x^\alpha D^\beta f\| \le C^{N+1} N^N \quad \textit{for every} \quad (\alpha, \beta) \in \mathcal{M}_N. \tag{2.4}$$

where

$$\mathcal{M}_N := \{(\alpha, \beta) : |\alpha| = N, |\beta| \le N \quad \textit{or} \quad |\alpha| \le N, |\beta| = N\}. \tag{2.5}$$

Proposition 2.2. *The following inclusions hold true:*

$$\bigcup_{0<\rho<1} S_\rho^{1-\rho}(\mathbb{R}^n) \subset S_1^{1\star}(\mathbb{R}^n) \subset S_1^1(\mathbb{R}^n).$$

Proof. Concerning the right inclusion, we first observe that (2.4) is equivalent to

$$\|x^\alpha D^\beta f\| \le C_0^{N+1} N! \quad \textit{for every} \quad (\alpha, \beta) \in \mathcal{M}_N, \tag{2.6}$$

for a new constant $C_0 > 0$. Since $(\alpha, \beta) \in \mathcal{M}_L$ with $L = \max\{|\alpha|, |\beta|\} \le |\alpha| + |\beta|$, then $f \in S_1^{1\star}(\mathbb{R}^n)$ implies for $C_0 > 1$

$$\|x^\alpha D^\beta f\| \le C_0^{|\alpha|+|\beta|+1} (|\alpha| + |\beta|)! \le C_1^{|\alpha|+|\beta|+1} \alpha! \beta! \tag{2.7}$$

for a new constant $C_1 > 0$. From (2.7), we deduce that $f \in S_1^1(\mathbb{R}^n)$ by means of embedding Sobolev estimates. The left inclusion easily follows by standard factorial estimates. $\square$

Example 1. The function $e^{-\langle x\rangle}$ belongs to $S_1^{1\star}(\mathbb{R}^n)$ but not to $\bigcup_{0<\rho<1} S_\rho^{1-\rho}(\mathbb{R}^n)$. The first assertion follows by a Faà di Bruno type formula

$$x^\alpha\partial^\beta e^{-\langle x\rangle} = x^\alpha e^{-\langle x\rangle}\sum_{j=1}^{|\beta|}(j!)^{-1}\sum_{\substack{\beta_1+\dots+\beta_j=\beta\\ |\beta_1|\geq 1,\dots,|\beta_j|\geq 1}}\frac{\beta!}{\beta_1!\dots\beta_j!}(\partial^{\beta_1}\langle x\rangle)\dots(\partial^{\beta_j}\langle x\rangle),$$

the estimate $|\partial^\beta\langle x\rangle|\leq C^{|\beta|+1}\beta!\langle x\rangle^{1-|\beta|}$ and standard factorial estimates. Clearly $e^{-\langle x\rangle}$ does not extend to an entire function in $\mathbb{C}^n$ while $S_\rho^{1-\rho}(\mathbb{R}^n)$ is contained in the set of all entire functions in $\mathbb{C}^n$, hence we get $e^{-\langle x\rangle}\notin S_\rho^{1-\rho}(\mathbb{R}^n)$ for any $\rho\in(0,1)$, as also evident from (1.3).

Theorem 2.3. *Assume that P in* (2.1) *is globally elliptic, i.e.,* (2.2) *is satisfied. If $u\in\mathcal{S}'(\mathbb{R}^n)$ is a solution of $Pu=f$ with f in $S_1^{1\star}(\mathbb{R}^n)$, then also $u\in S_1^{1\star}(\mathbb{R}^n)$. In particular, $Pu=0, u\in\mathcal{S}'(\mathbb{R}^n)$, implies u is in $S_1^{1\star}(\mathbb{R}^n)$.*

As a consequence of (1.3), Proposition 2.2, Theorem 2.3 and the fact that SG-ellipticity of P is invariant under linear perturbations, we deduce the following result on the eigenfunctions of P.

Corollary 2.4. *Under the previous assumptions on P, if $u\in\mathcal{S}'(\mathbb{R}^n)$ is a solution of $Pu=\lambda u$, for some $\lambda\in\mathbb{C}$, then for suitable constants $C>0,\varepsilon>0$:*

$$|\partial_x^\beta u(x)|\leq C^{|\beta|+1}\beta! e^{-\varepsilon|x|},\quad x\in\mathbb{R}^n,\beta\in\mathbb{Z}_+^n. \tag{2.8}$$

Proof of Theorem 2.3. As we observed, it is known that $Pu=f\in\mathcal{S}(\mathbb{R}^n), u$ in $\mathcal{S}'(\mathbb{R}^n)$ imply $u\in\mathcal{S}(\mathbb{R}^n)$. Then, choosing $C>1$ sufficiently large we may write for $N\leq m$

$$\|x^\alpha D^\beta u\|\leq C^{N+1}N^N \quad \textit{for every}\quad (\alpha,\beta)\in\mathcal{M}_N. \tag{2.9}$$

By induction, assume that (2.9) is valid for $N<M$, $M>m$, and prove it for $N=M$. For $(\alpha,\beta)\in\mathcal{M}_M$ we write

$$\|x^\alpha D^\beta u\| = \|x^{\alpha-\gamma}x^\gamma D^{\beta-\delta}D^\delta u\|$$

where we choose $\gamma\leq\alpha,\delta\leq\beta$ so that $(\gamma,\delta)\in\mathcal{M}_{M-m}$ and $(\alpha-\gamma,\beta-\delta)\in\mathcal{M}_m$. We use here the obvious vector sum property of the sets in (2.5):

$$\mathcal{M}_M\subset\mathcal{M}_{M-m}+\mathcal{M}_m.$$

Note that $|\alpha-\gamma|\leq m, |\beta-\delta|\leq m$. Then, by (2.3), we have

$$\begin{aligned}\|x^\alpha D^\beta u\| &\leq \|x^{\alpha-\gamma}D^{\beta-\delta}(x^\gamma D^\delta u)\| + \|x^{\alpha-\gamma}[x^\gamma, D^{\beta-\delta}]D^\delta u\|\\ &\leq C^\star\left(\|P(x^\gamma D^\delta u)\| + \|x^\gamma D^\delta u\|\right) + \|x^{\alpha-\gamma}[x^\gamma, D^{\beta-\delta}]D^\delta u\|\\ &\leq C^\star\left(\|x^\gamma D^\delta(Pu)\| + \|[P, x^\gamma D^\delta]u\| + \|x^\gamma D^\delta u\|\right) + \|x^{\alpha-\gamma}[x^\gamma, D^{\beta-\delta}]D^\delta u\|.\end{aligned}$$

Since $Pu=f\in S_1^{1\star}(\mathbb{R}^n)$ and $(\gamma,\delta)\in\mathcal{M}_{M-m}$, we have for some constant $C_1>1$

$$\|x^\alpha D^\delta(Pu)\|\leq C_1^{M-m+1}(M-m)^{M-m}\leq C_1^{M+1}M^M.$$

Write explicitly, by using (2.1):

$$[P, x^\gamma D^\delta] = \sum_{\substack{|\tilde\alpha|\le m\\ |\tilde\beta|\le m}} c_{\tilde\alpha\tilde\beta}[x^{\tilde\alpha} D^{\tilde\beta}, x^\gamma D^\delta].$$

Therefore, for $C_2 > 0$ such that $|c_{\tilde\alpha\tilde\beta}| \le C_2, |\tilde\alpha| \le m, |\tilde\beta| \le m$, we have

$$\|x^\alpha D^\beta u\| \le C^\star C_1^{M+1} M^M + C^\star C_2 \sum_{\substack{|\tilde\alpha|\le m\\ |\tilde\beta|\le m}} \|[x^{\tilde\alpha} D^{\tilde\beta}, x^\gamma D^\delta]u\| \qquad (2.10)$$

$$+ C^\star \|x^\gamma D^\delta u\| + \|x^{\alpha-\gamma}[x^\gamma, D^{\beta-\delta}]D^\delta u\|.$$

In (2.10), let us develop

$$[x^{\tilde\alpha} D^{\tilde\beta}, x^\gamma D^\delta] = \sum_{0\ne\sigma\le\tilde\beta, \sigma\le\gamma} c^1_{\tilde\alpha\tilde\beta\gamma\delta\sigma} x^{\gamma+\tilde\alpha-\sigma} D^{\delta+\tilde\beta-\sigma}$$
$$- \sum_{0\ne\sigma\le\tilde\alpha, \sigma\le\delta} c^2_{\tilde\alpha\tilde\beta\gamma\delta\sigma} x^{\gamma+\tilde\alpha-\sigma} D^{\delta+\tilde\beta-\sigma},$$

where

$$c^1_{\tilde\alpha\tilde\beta\gamma\delta\sigma} = \frac{(-i)^{|\sigma|}}{\sigma!} \frac{\tilde\beta!}{(\tilde\beta-\sigma)!} \frac{\gamma!}{(\gamma-\sigma)!}, \quad c^2_{\tilde\alpha\tilde\beta\gamma\delta\sigma} = \frac{(-i)^{|\sigma|}}{\sigma!} \frac{\tilde\alpha!}{(\tilde\alpha-\sigma)!} \frac{\delta!}{(\delta-\sigma)!}.$$

We observe that

$$|c^j_{\tilde\alpha\tilde\beta\gamma\delta\sigma}| \le C_3 M^{|\sigma|}, \quad j = 1, 2,$$

where the constant C_3 depends only on the order m and the dimension n. Therefore

$$\|[x^{\tilde\alpha} D^{\tilde\beta}, x^\gamma D^\delta]u\| \le C_3 \sum_{0\ne\sigma\le\tilde\beta, \sigma\le\gamma} + \sum_{0\ne\sigma\le\tilde\alpha, \sigma\le\delta} M^{|\sigma|} \|x^{\gamma+\tilde\alpha-\sigma} D^{\delta+\tilde\beta-\sigma} u\|. \quad (2.11)$$

Observe that $(\gamma, \delta) \in \mathcal{M}_{M-m}$ and $(\tilde\alpha, \tilde\beta) \in \mathcal{M}_{m'}$, for some $m', 0 \le m' \le m$, imply $(\gamma + \tilde\alpha, \delta + \tilde\beta) \in \mathcal{M}_{M'}$ for some $M' \le M$, and therefore in the sums above

$$(\gamma + \tilde\alpha - \sigma, \delta + \tilde\beta - \sigma) \in \mathcal{M}_L \quad \textit{for some} \quad L \le M - |\sigma|.$$

Then, from the inductive assumption, we have

$$\|x^{\gamma+\tilde\alpha-\sigma} D^{\delta+\tilde\beta-\sigma} u\| \le C^{L+1} L^L \le C^{M-|\sigma|+1} M^{M-|\sigma|}. \qquad (2.12)$$

Inserting (2.12) in (2.11) and denoting by C_4 the number of the terms in the sums, which depends only on m and on the dimension n, we conclude that

$$\|[x^{\tilde\alpha} D^{\tilde\beta}, x^\gamma D^\delta]u\| \le C_3 C_4 C^M M^M, \qquad (2.13)$$

since $C^{M-|\sigma|+1} \le C^M$ for $\sigma \ne 0$. Arguing similarly and observing that $|\beta - \delta| \le m$, we may estimate the last term in the right-hand side of (2.10) as follows:

$$\|x^{\alpha-\gamma}[x^\gamma, D^{\beta-\delta}]D^\delta u\| \le C_5 C^M M^M. \qquad (2.14)$$

In (2.10) we also have

$$\|x^\gamma D^\delta u\| \le C^{M-m+1}(M-m)^{M-m} \le C^M M^M. \qquad (2.15)$$

Inserting (2.13), (2.14), (2.15) in (2.10) and denoting by C_6 the number of the terms in the sum, with $|\tilde{\alpha}| \leq m, |\tilde{\beta}| \leq m$, finally we get

$$\|x^\alpha D^\beta u\| \leq C^\star C_1^{M+1} M^M + C^\star C_2 C_3 C_4 C_6 C^M M^M + C^\star C^M M^M + C_5 C^M M^M.$$

Hence, clearly for $C \geq C^\star C_2 C_3 C_4 C_6 + C^\star C_1 + C^\star + C_5$ we obtain the conclusion

$$\|x^\alpha D^\beta u\| \leq C^{M+1} M^M \quad \textit{for} \quad (\alpha, \beta) \in \mathcal{M}_M. \qquad \square$$

Remark 2.5. The result in Theorem 2.3 can be proved more generally for SG-elliptic operators of the form

$$P = \sum_{\substack{|\alpha| \leq m \\ |\beta| \leq m'}} c_{\alpha\beta} x^\alpha D^\beta$$

with $m > 0, m' > 0$, without essential changes in the proof except a slight modification in the definition of the space $S_1^{1\star}(\mathbb{R}^n)$. Moreover, we observe that in the statement of Theorem 2.3, we may take as starting assumption $u \in S_\mu^{\mu'}(\mathbb{R}^n), \mu > 1$, as it follows easily from the results in [5].

We conclude with an example representative of our results.

Example 2. Consider the operator

$$H = (1 + |x|^2)(-\Delta + 1) + L_1(x, D) \tag{2.16}$$

where $L_1(x, D)$ is a first order operator with polynomial coefficients of degree 1. H is elliptic according to (2.2). By Theorem 2.3, if u is a solution of the equation $Hu = f \in S_1^{1\star}(\mathbb{R}^n)$ and $u \in \mathcal{S}'(\mathbb{R}^n)$, then $u \in S_1^{1\star}(\mathbb{R}^n)$. In particular, cf. Corollary 2.4, we have exponential decay for the solutions of the homogeneous equation $Hu = 0, u \in \mathcal{S}'(\mathbb{R}^n)$. We may test more precisely the behavior at infinity of u in the one-dimensional case as follows. Consider in particular

$$Hu = -(1 + x^2)u'' + x^2 u - 2xu', \qquad x \in \mathbb{R}. \tag{2.17}$$

The operator H is L^2-self-adjoint and then there exists a sequence $\lambda_j \in \mathbb{R}, j = 1, 2, \ldots$, such that $Hu_j = \lambda_j u_j$ for some non-trivial $u_j \in \mathcal{S}(\mathbb{R})$, cf. [12]. From Corollary 2.4, we obtain (2.8). On the other hand, by the theory of asymptotic integration, see Tricomi [21], Wasow [22], we have

$$u_j(x) = Cx^{-1} e^{-|x|} + O(x^{-2} e^{-|x|}) \qquad \textit{when} \qquad |x| \to +\infty.$$

This shows that our result in Corollary 2.4 is sharp, in the sense that we cannot get (1.3) for $\mu < 1$.

3. Semilinear perturbations

The aim of this section is to outline results for exponential decay and uniform analyticity of solutions for semilinear perturbations of P.

Remark 3.1. We point out that the theorem below can be regarded as a generalization of the decay and analyticity of travelling waves for semilinear evolution PDEs in the framework of semilinear elliptic SG-equations without requiring a priori decay to zero for $|x| \to +\infty$ as in [3, 4].

We consider the semilinear model equation

$$Pu = (1+|x|^2)(-\Delta + 1)u + cu = F(u) + f(x), \qquad f \in S_1^1(\mathbb{R}^n), \tag{3.1}$$

where $c \in \mathbb{C}$ and the nonlinear term is polynomial

$$F(u) = \sum_{j=2}^{d} F_j u^j, \qquad F_j \in \mathbb{C}. \tag{3.2}$$

Here is the second main result.

Theorem 3.2. *If $u \in H^s(\mathbb{R}^n)$, for some $s > n/2$, is a solution of* (3.1) *with f in $S_1^1(\mathbb{R}^n)$, then also $u \in S_1^1(\mathbb{R}^n)$.*

Proof. We introduce, following [3, 9], H^s based norms defining $S_1^1(\mathbb{R}^n)$

$$|||v|||_{\varepsilon,\delta} = \sum_{\alpha,\beta \in \mathbb{Z}_+^n} \|\{v\}_{\varepsilon,\delta}^{\alpha,\beta}\|_s, \tag{3.3}$$

where

$$\{v\}_{\varepsilon,\delta}^{\alpha,\beta}(x) = \frac{\varepsilon^{|\alpha|}\delta^{|\beta|}}{\alpha!\beta!} x^\alpha \partial_x^\beta v(x) \tag{3.4}$$

for $\alpha, \beta \in \mathbb{Z}_+^n$. Equivalence with the definition from (1.2) is easily proved by embedding Sobolev estimates. We define also the partial sums

$$S_N^{\varepsilon,\delta}[v] = \sum_{\substack{\alpha,\beta \in \mathbb{Z}_+^n \\ |\alpha|+|\beta| \le N}} \|\{v\}_{\varepsilon,\delta}^{\alpha,\beta}\|_s, \quad N \in \mathbb{Z}_+. \tag{3.5}$$

Clearly

$$\lim_{N\to\infty} S_N^{\varepsilon,\delta}[v] = \sup_{N \in \mathbb{Z}_+} S_N^{\varepsilon,\delta}[v] = |||v|||_{\varepsilon,\delta}. \tag{3.6}$$

As $F(u)$ is polynomial, we may assume without loss of generality that $F(u) = u^k$ for some $k \in \mathbb{Z}_+$, $k \ge 2$. In view of the results in [6, 7, 15, 19] on pseudodifferential calculus, we can find $\lambda_0 \in \mathbb{C}$ such that $-\lambda_0 \notin \operatorname{spec}(P)$ and we get that for every $s > 0$ the operator

$$(P+\lambda_0)^{-1} \circ x^\mu \partial_x^\nu : H^s(\mathbb{R}^n) \mapsto H^s(\mathbb{R}^n) \tag{3.7}$$

is continuous for all $\mu, \nu \in \mathbb{Z}_+^n, |\mu| \le 2, |\nu| \le 2$. We can rewrite (3.1) as follows

$$P_0 u := Pu + \lambda_0 u = \lambda_0 u + u^k + f(x) \tag{3.8}$$

Next, we show identities for commutators. With respect to Section 2, we need here a somewhat different expression, namely: there exist $\kappa_{\mu\nu} \in \mathbb{R}$, $\mu, \nu \in \mathbb{Z}_+$, $1 \le |\mu| + |\nu| \le 3$, such that

$$[x^\alpha \partial_x^\beta, |x|^2 \Delta] u = \sum_{\substack{\mu \le \alpha, \nu \le \beta \\ 1 \le |\mu|+|\nu| \le 3}} \kappa_{\mu\nu} \binom{\alpha}{\mu} \binom{\beta}{\nu} x^\mu \partial_x^\nu (x^{\alpha-\mu} \partial_x^{\beta-\nu} u) \tag{3.9}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$, $|\alpha| + |\beta| > 0$.

The validity of (3.9) follows from the following commutator identities for two variables $y, z \in \mathbb{R}$:

$$z^\sigma \partial_y^\rho (y^2 \partial_z^2 u) = z^\sigma \partial_z^2 \left(y^2 \partial_y^\rho + 2\rho y \partial_y^{\rho-1} u + \rho(\rho-1) \partial_y^{\rho-2} u \right) \tag{3.10}$$

$$z^\sigma \partial_z^2 v = \partial_z^2 (z^\sigma v) - 2\sigma \partial_z (z^\sigma v) - \sigma(\sigma-1) z^{\sigma-2} v \tag{3.11}$$

for all $\rho, \sigma \in \mathbb{Z}_+$. By (3.10) and (3.11) we obtain

$$\begin{aligned}
[z^\sigma \partial_y^\rho, y^2 \partial_z^2] u = {} & 2\rho y \partial_z^2 (z^\sigma \partial_y^{\rho-1} u) + \rho(\rho-1) \partial_z^2 (z^\sigma \partial_y^{\rho-2} u) \\
& -2\sigma y^2 \partial_z (z^{\sigma-1} \partial_y^\rho u) - 4\rho\sigma y \partial_z (z^{\sigma-1} \partial_y^{\rho-1} u) \\
& -2\rho(\rho-1)\sigma\rho \partial_z (z^{\sigma-1} \partial_y^{\rho-2} u) \\
& -\sigma(\sigma-1) y^2 (z^{\sigma-1} \partial_y^{\rho-2} u) - 2\rho\sigma(\sigma-1) y (z^{\sigma-2} \partial_y^{\rho-1} u) \\
& -\rho(\rho-1)\sigma(\sigma-1)(z^{\sigma-2} \partial_y^{\rho-2} u)
\end{aligned}$$

for all $\rho, \sigma \in \mathbb{Z}_+$ and, in view of standard combinatorial identities, we derive (3.9). Similar identities can be found for the commutators $[x^\beta \partial_x^\alpha, \Delta]$ and $[x^\beta \partial_x^\alpha, |x|^2]$. Next, we observe that $P_0 u = g$ implies

$$\{u\}_{\varepsilon,\delta}^{\alpha,\beta} = -\frac{\varepsilon^{|\alpha|} \delta^{|\beta|}}{\alpha! \beta!} P_0^{-1} ([x^\alpha \partial_x^\beta, P_0] u) + P_0^{-1} \{g\}_{\varepsilon,\delta}^{\alpha,\beta}$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$. Therefore, taking into account (3.9), we derive

$$\begin{aligned}
\{u\}_{\varepsilon,\delta}^{\alpha,\beta} = {} & \sum_{\substack{\mu \le \alpha, \nu \le \beta \\ 1 \le |\mu|+|\nu| \le 3}} \tilde{\kappa}_{\mu\nu} \binom{\alpha}{\mu} \binom{\beta}{\nu} \frac{\varepsilon^{|\mu|} \delta^{|\nu|}}{\mu! \nu!} (P_0^{-1} \circ x^\mu \partial_x^\nu) \{u\}_{\varepsilon,\delta}^{\alpha-\mu,\beta-\nu} \\
& + \lambda_0 P_0^{-1} (\{u\}_{\varepsilon,\delta}^{\alpha,\beta}) + P_0^{-1} (\{u^k\}_{\varepsilon,\delta}^{\alpha,\beta}) + P_0^{-1} \{f\}_{\varepsilon,\delta}^{\alpha,\beta}.
\end{aligned} \tag{3.12}$$

Now we use the smoothing property (3.7) and obtain that for some $C_0 > 0$

$$\| \lambda_0 P_0^{-1} (\{u\}_{\varepsilon,\delta}^{\alpha,\beta}) \|_s \le C_0 \frac{\varepsilon}{\alpha_j} \| \{u\}_{\varepsilon,\delta}^{\alpha-e_j,\beta} \|_s \tag{3.13}$$

if $\alpha_j \ge 1$ for some $j \in \{1, \ldots, n\}$, and

$$\| \lambda_0 P_0^{-1} (\{u\}_{\varepsilon,\delta}^{\alpha,\beta}) \|_s \le C_0 \frac{\delta}{\beta_\ell} \| \{u\}_{\varepsilon,\delta}^{\alpha,\beta-e_\ell} \|_s \tag{3.14}$$

if $\beta_\ell \geq 1$. Similarly, from (3.5) and (3.7), we deduce that

$$\sum_{\substack{\alpha,\beta\in\mathbb{Z}^n_+\\ |\alpha|+|\beta|\leq N}} \sum_{\substack{\mu\leq\alpha,\nu\leq\beta\\ 1\leq|\mu|+|\nu|\leq 3}} \tilde{\kappa}_{\mu\nu}\binom{\alpha}{\mu}\binom{\beta}{\nu}\frac{\varepsilon^{|\mu|}\delta^{|\nu|}}{\mu!\nu!}\|(P_0^{-1}\circ x^\mu\partial_x^\nu)\{u\}^{\alpha-\mu,\beta-\nu}_{\varepsilon,\delta}\|_s \qquad (3.15)$$
$$\leq C_1(\delta+\varepsilon)S^{\varepsilon,\delta}_{N-1}[u].$$

The identity (3.12), the estimates (3.13), (3.14), (3.15), the Banach algebra properties of the norms (cf. [3, 9]) yield that there exists a constant $C > 0$ such that

$$S^{\varepsilon,\delta}_N[u] \leq C\|u\|_s + C(\delta+\varepsilon)S^{\varepsilon,\delta}_{N-1}[u] + C(\delta+\varepsilon)(S^{\varepsilon,\delta}_{N-1}[u])^k + |||f|||_{\varepsilon,\delta} \qquad (3.16)$$

for all $N = 1, 2, \ldots$, $\varepsilon, \delta \in (0,1)$. Choosing δ and ε small enough, (3.16) leads to (3.6), which concludes the proof. $\square$

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A Short Description of Kinetic Models for Chemotaxis

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Abstract. We describe how the Keller-Segel model can be obtained as a drift-diffusion limit of kinetic models. Three different examples with global kinetic solutions yield different chemotactical sensitivity functions, including the case of a constant coefficient, where blow up in the limit may occur, the case with density threshold and an intermediate case for which the corresponding perturbed Keller-Segel models have global solutions.

1. Introduction

The amoeba *Dictyostelium discoideum* has a complex social behavior that has long attracted the attention of scientists from different fields. For mathematicians, the most widely used model is the Keller-Segel model (see [14, 15]; for an earlier version, see [20]). This model describes a population of cells moving toward higher concentrations of a certain chemical substances produced by themselves. It was derived from Fick's law, namely, by considering currents respectively for the cell and the chemo-attractant concentrations $\rho(x,t)$ and $S(x,t)$ defined in $(x,t) \in \Omega \times \mathbb{R}_+$, $\Omega \subset \mathbb{R}^2$, given by

$$\begin{aligned} J_\rho &:= \kappa_1 \nabla S - \kappa_2 \nabla \rho \ , \\ J_S &:= -\kappa_3 \nabla S \ , \end{aligned}$$

and associated to the conservation of mass

$$\partial_t \phi = \nabla \cdot J_\phi + \mathcal{Q}_\phi \ ,$$

where $\mathcal{Q}_\rho = 0$ and $\mathcal{Q}_S = \rho$ are the production/destruction terms for ρ and S. Considering certain normalizations and the limit of high diffusion (where the diffusion of the chemical substance is considered much higher than the diffusion of cells), the simplified version of the Keller-Segel model is given by:

$$\partial_t \rho = \nabla \cdot (\nabla \rho - \chi(S,\rho)\rho\nabla S) \ , \tag{1.1}$$

$$\delta\partial_t S - \Delta S = \rho \ , \tag{1.2}$$

$$\rho(\cdot,0) = \rho^{\mathrm{I}} \ , \tag{1.3}$$

where $\rho(x,t) \geq 0$ and $S(x,t) \geq 0$ if $\rho^{\mathrm{I}} \geq 0$, $\delta \in \{0,1\}$, satisfy suitable decay conditions at infinity (or Neumann boundary conditions on the border, for bounded Ω). χ is called the chemotactical sensitivity.

These equations (and some of their generalizations) have attracted much attention. In particular, the exhibition of precise conditions such that their solutions exist globally or present finite-time-blow-up is an important mathematical question. In [13], for the case $\delta = 0$, it was proved the existence of values $C(\Omega)$ and $C^*(\Omega)$ such that, for the conserved total mass $M = \int_\Omega \rho^{\mathrm{I}} \mathrm{d}x$, if $\chi M < C(\Omega)$ or $\chi M > C^*(\Omega)$, then solutions exist globally or present finite-time-blow-up, respectively.

In [7] some of the blow up profiles were described as Dirac-delta type concentrations. In [16] it was proved that $C(\Omega) \geq 4\pi$ and, in particular, for radial solutions, $C(B_R) = 8\pi$, where $B_R \subset \mathbb{R}^2$ is a ball centered in the origin with arbitrary radius R. Later on, in [17], it was proved for general, but bounded Ω, that if there is a blow up point, then $\chi M \geq 8\pi$ if it is in the interior of Ω, or $8\pi > \chi M \geq 4\pi$ if the blow-up occurs on the border $\partial\Omega$.

For the whole space, with $\delta = 0$, the problem was solved in [5], where it was proved that $C(\mathbb{R}^2) = C^*(\mathbb{R}^2) = 8\pi$. For bounded domains, this problem is still open (see [6]). For more detailed reviews on chemotaxis, see also [10, 11].

The Keller-Segel model was first obtained from the phenomenological viewpoint. Its derivation from more basic principles was obtained in [21], as the limit dynamics of moderately interacting stochastic many-particle systems. A second approach was introduced in [19, 9] where the Keller-Segel model was formally obtained from kinetic models for chemotaxis, introduced in [1, 18]. The rigorous derivation of the Keller-Segel model from kinetic models was given in [3] for the case $\delta = 0$ and $\Omega = \mathbb{R}^3$ and were generalized in [12].

Here we consider only $\Omega = \mathbb{R}^2$, despite the fact that all theorems can be generalized (with minor modifications) to the case $\Omega = \mathbb{R}^3$. For bounded domains, there are no results available. We also assume, for simplicity, $\delta = 0$, but similar results hold for $\delta = 1$. See [4, 12].

The kinetic models for chemotaxis consist in a transport equation for the phase space cell density, i.e., $f(x,v,t)$, where $v \in V$ is the cell velocity, for a compact and spherically symmetric set of all possible velocities $V \subset \mathbb{R}^2$. Given a turning kernel, $T[S,\rho](x,v,v',t)$, the rate of changing from velocity v' to velocity v, in a space-time point $(x,t) \in \Omega \times V$ in the presence of chemical substance S

and cell concentration ρ, we have

$$\partial_t f(x,v,t) + v \cdot \nabla f(x,v,t) = \mathcal{T}[S,\rho](f)(x,v,t) \ , \tag{1.4}$$

where

$$\mathcal{T}[S,\rho](f)(x,v,t) := \int_V (T[S,\rho](x,v,v',t)f(x,v',t) - T[S,\rho](x,v',v,t)f(x,v,t))\,\mathrm{d}v'.$$

This equation should be coupled with Equation (1.2), where

$$\rho(x,t) = \int_V f(x,v,t)\mathrm{d}v \ . \tag{1.5}$$

Initial conditions are given by

$$f(x,v,0) = f^{\mathrm{I}}(x,v) \geq 0 \ . \tag{1.6}$$

We simplify our notation putting $f = f(x,v,t)$, $f' = f(x,v',t)$, $T[S,\rho] = T[S,\rho](x,v,v',t)$ and $T^*[S,\rho] = T[S,\rho](x,v',v,t)$.

2. Formal and rigorous convergence

Going back to the Othmer-Dunbar-Alt model (1.4–1.6) and (1.2) and considering typical values for all the variables involved, we re-scale the problem in the new variables $\bar{x} = x/x_0$, $\bar{t} = t/t_0$, $\bar{v} = v/v_0$, $\bar{f} = f/f_0$, $\bar{S} = S/S_0$, $\bar{T} = T/T_0$, $\bar{\rho} = \rho/\rho_0$. Then, equations (1.4–1.6) and (1.2) are modified to (we simplify the notation, dropping all bars)

$$\partial_t f + \frac{v_0}{x_0/t_0} v \cdot \nabla f = T_0 t_0 v_0^n \int_V (T[S,\rho]f' - T^*[S,\rho]f)\,\mathrm{d}v' \ , \tag{2.1}$$

$$\frac{\delta}{S_0}\partial_t S - \frac{t_0}{x_0^2}\Delta S = \frac{t_0\rho_0}{S_0}\rho \ , \tag{2.2}$$

$$\rho = \frac{f_0 v_0^n}{\rho_0}\int_V f\mathrm{d}v \ . \tag{2.3}$$

Now, we consider that $t_0 = x_0^2$ (*diffusive scaling*) and the microscopic typical velocity v_0 is much larger than the typical macroscopic velocity x_0/t_0, i.e.,

$$\varepsilon := \frac{x_0/t_0}{v_0} \ll 1 \ .$$

We impose that the collisional term is very strong, actually, of order ε^{-2}. We finally assume some normalizations, impose $\delta = 0$, so that the system (2.1–2.3) becomes

$$\varepsilon^2 \partial_t f_\varepsilon + \varepsilon v \cdot \nabla f_\varepsilon = \int_V (T_\varepsilon[S_\varepsilon,\rho_\varepsilon]f'_\varepsilon - T^*_\varepsilon[S_\varepsilon,f_\varepsilon]f_\varepsilon)\,\mathrm{d}v' \tag{2.4}$$

$$-\Delta S_\varepsilon = \rho_\varepsilon \ , \tag{2.5}$$

$$\rho_\varepsilon = \int_V f_\varepsilon \mathrm{d}v \ . \tag{2.6}$$

We now look for the drift-diffusion limit of the above model. Namely, we look for a set of equation such that its solution is a good approximation, for small ε, of the functions ρ_ε and S_ε (which are the macroscopically relevant variables).

We start by considering the (formal) expansions

$$f_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots \qquad \text{and} \qquad S_\varepsilon = S_0 + \varepsilon S_1 + \varepsilon^2 S_2 + \cdots \tag{2.7}$$

We also assume that, for the turning kernel, the expansion

$$T_\varepsilon[S,\rho] = T_0[S,\rho] + \varepsilon T_1[S,\rho] + \cdots \tag{2.8}$$

is well defined and that

(A0) There exists a bounded velocity distribution $F(v) > 0$, independent of x, t, and S, such that the detailed balance principle $T_0^*[S]F = T_0[S]F'$ holds. The flow produced by this equilibrium distribution vanishes, and F is normalized:

$$\int_V vF(v)\,\mathrm{d}v = 0\,, \qquad \int_V F(v)\,\mathrm{d}v = 1\,. \tag{2.9}$$

The turning rate $T_0[S]$ is bounded, and there exists a constant $\gamma > 0$ such that $T_0[S]/F \geq \gamma$, $\forall\,(v,v') \in V\times V$, $x\in\mathbb{R}^3$, $t>0$.

We put expansions (2.7) and (2.8) in the system (2.4–2.6) and match terms of the same order of ε.

In order ε^0, we find

$$0 = \int_V \left(T_0[S_0,\rho_0]f_0 - T_0^*[S_0,\rho_0]f_0'\right)\mathrm{d}v'\ , \tag{2.10}$$

$$-\Delta S_0 = \rho_0 := \int_V f_0 \mathrm{d}v\ . \tag{2.11}$$

From the equation

$$\iint_{V\times V}\left(T_0[S_0,\rho_0]f_0' - T_0^*[S_0,\rho_0]f_0\right)\frac{f_0}{F}\mathrm{d}v'\mathrm{d}v$$

$$= \frac{1}{2}\iint_{V\times V} T_0[S_0,\rho_0]F'\left(\frac{f_0}{F} - \frac{f_0'}{F'}\right)^2 \mathrm{d}v\mathrm{d}v'$$

we deduce that the solution of Equation (2.10–2.11) is given by $f_0(x,v,t) = \rho_0(x,t)F(v)$ and $S_0(x,t) = (2\pi)^{-1}\int_V \log|x-y|\rho_0(y,t)\mathrm{d}y$, where $\rho_0(x,t)$ is the unknown macroscopic density.

Now, we go back to the System (2.4–2.6) and isolate terms of order ε^1

$$v\cdot\nabla f_0 = \int_V \left(\mathcal{T}_1[S_0,\rho_0](f_0) + \mathcal{T}_0[S_0,\rho_0](f_1)\right)\mathrm{d}v'\ , \tag{2.12}$$

$$-\Delta S_1 = \rho_1 := \int_V f_1 \mathrm{d}v\ . \tag{2.13}$$

with

$$\mathcal{T}_k[S,\rho](f) := \int_V \left(T_k[S,\rho]f' - T_k^*[S,\rho]f\right) \mathrm{d}v' \ , \quad k = 0,1 \ . \tag{2.14}$$

Equation (2.12) can be solved with help of Lemma 2 of [3] so that

$$\begin{aligned} f_1(x,v,t) = & -\kappa[S_0,\rho_0](x,v,t) \cdot \nabla\rho_0(x,t) \\ & -\Theta[S_0,\rho_0](x,v,t)\rho_0(x,t) + \rho_1(x,t)F(v) \,, \end{aligned} \tag{2.15}$$

where κ and Θ are the solutions of

$$\mathcal{T}_0[S_0,\rho_0](\kappa) = -vF \,, \tag{2.16}$$

$$\mathcal{T}_0[S_0,\rho_0](\Theta) = \mathcal{T}_1[S_0,\rho_0](F) \,, \tag{2.17}$$

and ρ_1, the macroscopic density of f_1, is a new unknown.

Back to the equations (2.4–2.6), after integrating order ε^2 terms, and using the previous results we find

$$\partial_t\rho_0 + \nabla\cdot\left(D(S_0,\rho_0)\nabla\rho_0 - \Gamma(S_0,\rho_0)\rho_0\right) = 0 \ , \tag{2.18}$$

where the diffusivity tensor and the convection vector are given by

$$D[S_0,\rho_0](x,t) = \int_V v\otimes\kappa[S_0,\rho_0](x,v,t)\mathrm{d}v \ , \tag{2.19}$$

$$\Gamma[S_0,\rho_0](x,t) = -\int_V v\Theta[S_0,\rho_0](x,v,t)\mathrm{d}v \ . \tag{2.20}$$

To finish the formal deduction, we only need to couple equation (2.18) to

$$-\Delta S_0 = \rho_0 \ . \tag{2.21}$$

Let us define the symmetric and anti-symmetric parts of $T_\varepsilon[S,\rho]F$, respectively, by:

$$\phi_\varepsilon^S[S,\rho] := \frac{T_\varepsilon[S,\rho]F' + T_\varepsilon^*[S,\rho]F}{2} \,, \tag{2.22}$$

$$\phi_\varepsilon^A[S,\rho] := \frac{T_\varepsilon[S,\rho]F' - T_\varepsilon^*[S,\rho]F}{2} \,. \tag{2.23}$$

Now, we are ready to state the *rigorous convergence results.* We will not prove them here, but the proofs can be found in references [3, 4, 12].

Theorem 2.1. *Let $F \in L^\infty(V)$ be a positive velocity distribution satisfying Assumption* (A0) *and let $\phi_\varepsilon^S[S]$ and $\phi_\varepsilon^A[S]$ be defined as in* (2.22) *and* (2.23). *Assume that there exist $q > 3$, $\lambda_0 > 0$, and a non-decreasing function $\Lambda \in L^\infty_{\mathrm{loc}}([0,\infty))$, such that*

$$\frac{f^{\mathrm{I}}}{F} \in \mathcal{X}_q := L^1_+ \cap L^q\left(\mathbb{R}^2 \times V;\ F\,\mathrm{d}x\,\mathrm{d}v\right) \,, \tag{2.24}$$

$$\phi_\varepsilon^S[S,\rho] \geq \lambda_0(1 - \varepsilon\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^2)}))FF' \,, \tag{2.25}$$

$$\int_V \frac{\phi_\varepsilon^A[S,\rho]^2}{F\phi_\varepsilon^S[S,\rho]}\mathrm{d}v' \leq \varepsilon^2\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^2)}) \,. \tag{2.26}$$

Then there exists $t^* > 0$, independent of ε, such that the existence time of the local mild solution of (2.4–2.6) *is larger than t^*, and the solution satisfies, uniformly in ε,*

$$\begin{aligned} &\frac{f_\varepsilon}{F} \in L^\infty(0, t^*;\ \mathcal{X}_q)\,, \\ &S_\varepsilon \in L^\infty(0, t^*;\ L^p \cap C^{1,\alpha}(\mathbb{R}^2))\,, \quad \alpha < \frac{q-2}{q}\ ,\ 3 < p < \infty \qquad (2.27)\\ r_\varepsilon = &\frac{f_\varepsilon - \rho_\varepsilon F}{\varepsilon} \in L^2\left(\mathbb{R}^2 \times V \times (0, t^*);\ \frac{\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t}{F}\right)\,. \end{aligned}$$

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *hold. Assume further that for any functions Σ_ε uniformly bounded (as $\varepsilon \to 0$) in $L^\infty_{\mathrm{loc}}(0, \infty;\ C^{1,\alpha}(\mathbb{R}^2))$ for some $0 < \alpha \leq 1$, such that Σ_ε and $\nabla\Sigma_\varepsilon$ converge to Σ_0 and $\nabla\Sigma_0$, respectively, in $L^p_{\mathrm{loc}}(\mathbb{R}^2 \times [0, \infty))$ for some $p > 3/2$ and η_ε converges to η_0 in $L^2_{\mathrm{loc}}(\mathbb{R}^2 \times [0, \infty))$, we have the convergence*

$$\begin{aligned} &T_\varepsilon[\Sigma_\varepsilon, \eta_\varepsilon] \to T_0[\Sigma_0, \eta_0] \qquad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2 \times V \times V \times [0, \infty))\,, \\ &\frac{\mathcal{T}_\varepsilon[\Sigma_\varepsilon, \eta_\varepsilon](F)}{\varepsilon} = \frac{2}{\varepsilon}\int_V \phi^A_\varepsilon[\Sigma_\varepsilon, \eta_\varepsilon]\mathrm{d}v' \to \mathcal{T}_1[\Sigma_0, \eta_0](F) \ \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^2 \times V \times [0, \infty)). \end{aligned}$$

Then, the solutions of (2.4–2.6) *satisfy (possibly after extracting subsequences)*

$$\begin{aligned} \rho_\varepsilon &\to \rho_0 \qquad \text{in } L^2_{\mathrm{loc}}(\mathbb{R}^2 \times (0, t^*))\ , \\ S_\varepsilon &\to S_0 \qquad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^2 \times (0, t^*))\,,\ 1 \leq q < \infty\,, \\ \nabla S_\varepsilon &\to \nabla S_0 \qquad \text{in } L^q_{\mathrm{loc}}(\mathbb{R}^2 \times (0, t^*))\,,\ 1 \leq q < \infty\,, \end{aligned}$$

and their limits are weak solutions of (2.18–2.21) *subject to the initial condition*

$$\begin{aligned} \rho_0(x, 0) &= \int_V f^{\mathrm{I}}(x, v)\,\mathrm{d}v\ , \\ S_0(x, 0) &= S^{\mathrm{I}}(x)\ . \end{aligned}$$

3. Models with global existence and their drift-diffusion limits

In this section, we give some particular examples of turning kernels for which it is possible to prove global existence of solutions. In some of these cases, it will be also possible to conclude bounds for the solution of the limit Keller-Segel models.

In this section we fix ε. We always assume $(A0)$. It is easy to see that all turning kernels obey the assumptions in Theorems 2.1 and 2.2. For Example 2, see Remark 3.1.

Example 1. [12] Let us suppose that there is a constant C such that

$$T_\varepsilon[S, \rho](x, v, v', t) \leq C\left(1 + S(x + \varepsilon v, t) + S(x - \varepsilon v, t)\right)\ .$$

Assume further that $f^{\mathrm{I}} \in L^1_+ \cap L^\infty(\mathbb{R}^2 \times V)$ Then, there is a global solution $f(\cdot,\cdot,\cdot) \in L^1_+ \cap L^\infty(\mathbb{R}^2 \times V)$ and $S(\cdot,t) \in L^p(\mathbb{R}^2)$, $p \in [2,\infty]$, $\forall t \in [0,\infty)$ of the system (2.4–2.6) for any fixed $\varepsilon > 0$.

If, in the previous example, we assume that $T_\varepsilon[S,\rho](x,v,v',t) = \psi(S(x,t), S(x+\varepsilon v,t))$, for a smooth function $\psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ then the coefficients in the limit equation are given by (see [3])

$$D[S_0,\rho_0] = \frac{1}{2\psi(S_0,S_0)} \int_V v^2 F(|v|)\mathrm{d}v \; ,$$
$$\Gamma[S_0,\rho_0] = \frac{\partial_2\psi(S_0,S_0)}{2\psi(S_0,S_0)} \int_V v^2 F(|v|)\mathrm{d}v \nabla S_0 \; ,$$

where ∂_2 denotes differentiation with respect to the second variable. If ψ is at most linear in the second variable, then global existence is guaranteed. If, moreover, $\psi(S,\tilde{S}) = \Psi(\tilde{S} - S) \le A\tilde{S} + B$ for positive constants A and B, then we have as limit model the Keller-Segel model with constant coefficients D and χ, which presents finite-time-blow-up for certain initial conditions.

Now, we consider a two-parameters turning kernel depending on the phase-space density:

$$T_{\varepsilon,\mu}[S,f](x,v,v',t) = \Phi(S(x + \varepsilon\zeta_\mu(f(x,v,t))v,t) - S(x,t))F(v) \; , \tag{3.1}$$

where

$$C_0(\mu) := \sup_{y\ge 0} \zeta_\mu(y)y \; , \quad \lim_{\mu\to 0} C_0(\mu) = \infty \; , \tag{3.2}$$

and for increasing function Φ such that $0 < \Phi_{\min} \le \Phi(y) \le Ay + B$, A and B positive constants and $\zeta_\mu : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and bounded for all $\mu \ge 0$.

Example 2. [2] For any fixed $\mu \ge 0$ and $\varepsilon > 0$ there exist global solutions of the kinetic model (2.4–2.6), i.e., for any $t > 0$, $f_{\varepsilon,\mu} \in L^\infty(0,t;L^\infty(\mathbb{R}^2 \times V))$ and $S_{\varepsilon,\mu} \in L^\infty(0,t;L^\infty(\mathbb{R}^2))$. Furthermore, $||\rho_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^2)}$ and $||S_{\varepsilon,\mu}(\cdot,t)||_{L^\infty(\mathbb{R}^2)}$ are bounded by μ-independent functions. For strictly positive μ we find as the drift-diffusion limit of this model the perturbation of the Keller-Segel model introduced in [22, 23], i.e., with constant diffusivity and sensitivity given by

$$\chi(S,\rho) = \frac{\Psi'(0)}{\Psi(0)} \int_V \zeta_\mu(\rho F(v))F(v)v^2\mathrm{d}v \; . \tag{3.3}$$

Remark 3.1. In order to extend Theorems 2.1 and 2.2 to the case where the turning kernel depends also on f, it is important to prove the convergence $||f_\varepsilon - f_0||_{L^p(\mathbb{R}^2\times V)} \to 0$, for some $p \in [1,\infty]$. This is a simple consequence of the convergence $\rho_\varepsilon \to \rho_0$ in $L^2(\mathbb{R}^2)$ and the boundedness of the remainder

$$r_\varepsilon := \frac{f_\varepsilon - \rho_\varepsilon F}{\varepsilon} \; .$$

For details, see [2]. It is important to note, that, for $\mu > 0$, it is possible to prove global convergence, i.e., the maximum time t^* in Theorem 2.2 can be arbitrarily extended.

In the first example the turning kernel depended only of S, while in the second case we introduced a dependence on the cell density such that for high concentrations the turning kernel becomes constant. Now, we consider a stronger assumption such that the chemotactical part $T_\varepsilon - T_0$ vanishes for densities above a certain strictly positive threshold $\bar{\rho}$.

Example 3. [4] Let us consider the turning kernel given by

$$T_\varepsilon[S,\rho](x,v,v',t) = \Psi(S(x+\varepsilon\zeta(\rho)v,t) - S(x,t))F(v)$$

such that there is an upper bound $\bar{\rho}$, i.e., $\zeta(\rho) = 0$ for $\rho \geq \bar{\rho}$ or let us consider a turning kernel

$$T_\varepsilon[S,\rho](x,v,v',t) = \lambda[S,\rho](x,t)F(v) + \varepsilon F(v)a(S,\rho)v\cdot\nabla S$$

for ε small enough and $a(S,\rho) = 0$ for $\rho \geq \bar{\rho}$. We also impose that

$$\sup_{\rho\geq 0, S\geq 0} \frac{a(S,\rho)}{\rho - \bar{\rho}} < \infty \quad \text{and} \quad \sup_{\rho\geq 0, S\geq 0} \frac{\zeta(\rho)}{\rho - \bar{\rho}} < \infty\ .$$

Let us suppose that initial conditions are given by $f^{\mathrm{I}}(x,v) = \rho^{\mathrm{I}}(x)F(v)$, $\rho^{\mathrm{I}} \in L^1_+ \cap L^\infty(\mathbb{R}^2)$, $S^{\mathrm{I}} = 0$. Then the solution (f,S) of the nonlinear system (2.4–2.5) exists globally: $f \in L^\infty(0,\infty; L^1_+ \cap L^\infty(\mathbb{R}^2\times V))$, $S \in L^\infty(0,t;L^p(\mathbb{R}^2))$, $p\in(1,\infty]$, $\forall t \in (0,\infty)$. Furthermore,

$$||\rho(\cdot,t)||_{L^\infty(\mathbb{R}^2)} \leq \left|\left|\frac{f(\cdot,\cdot,t)}{F}\right|\right|_{L^\infty(\mathbb{R}^2\times V)} \leq \max\{||\rho^{\mathrm{I}}||_{L^\infty(\mathbb{R}^2)}, \bar{\rho}\}\ ,\ \forall t\in\mathbb{R}_+\ .$$

As a direct consequence of the previous result (more specifically from the fact that the bound for $||\rho(\cdot,t)||_{L^\infty(\mathbb{R}^2)}$ is ε-independent), we reproduce the results in [8] (with some technical differences), namely the global existence of solution of Keller-Segel models with constant diffusivity and sensitivity such that $\chi(\rho) = 0$ for $\rho \geq \bar{\rho}$. For details, see [4].

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Hyperbolic Problems and Regularity Questions
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Eigenvalues, Eigenfunctions in Domains Becoming Unbounded

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Abstract. The aim of this work is to analyze the asymptotic behavior of the eigenmodes of some elliptic eigenvalue problems set on domains becoming unbounded in one or several directions.

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1. Introduction

In the modelization of physical phenomena, it often appears from the experience or heuristic arguments that the variations of a given quantity with respect to some variables is small. Then it is assumed that they vanish hence the quantity is independent of these variables which gives a simplified model in lower dimension. This approximation is suitable if the domain where the phenomenon takes place is large in some directions and if the external forces vary weakly in these directions. It is used, for instance in hydrodynamics (see [8, 10]) and in electromagnetics (see [9]).

The mathematical analysis of this approximation has been made for several differential equations or systems: see [2, 3, 4, 5, 6]. Let us show on an elementary example what kind of results is obtained in these papers. Let ℓ be a positive real number and $\Omega_\ell = (-\ell, \ell) \times (-1, 1)$. Denoting by (x_1, x_2) the points in $\mathbb{R}^2$, let $f = f(x_2) \in L^2((-1,1))$ and u_ℓ be the solution to

$$-\Delta u_\ell = f, \qquad \text{in } \Omega_\ell, \qquad u_\ell = 0 \qquad \text{on } \partial\Omega_\ell.$$

Then we can show (see [2, 4]) that for any given $\ell_0 > 0$,

$$u_\ell \to u_\infty \qquad \text{in } H_0^1(\Omega_{\ell_0}) = W_0^{1,2}(\Omega_{\ell_0}),$$

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where u_∞ is the solution to

$$-u_\infty'' = f \qquad \text{in } (-1,1), \qquad u_\infty = 0 \qquad \text{on } \{-1,1\}.$$

Besides, the convergence rate is greater than any power of $\frac{1}{\ell}$. More precisely, for all $r > 0$, there exists a constant C_r such that

$$||u_\ell - u_\infty||_{H^1(\Omega_{\ell_0})} \le \frac{C_r}{\ell^r} \qquad \forall \ell > 0.$$

See [2, 4] for exponential rate of convergence.

In this note, we will perform a similar analysis for "generalized" eigenvalue problems (see (2.5), (2.6)). We refer to [3] for results on "classical" eigenvalue problems. The rest of the article is organized as follows. In the next section, we set the eigenvalue problems and state our main results. Section 3 is devoted to their proofs.

2. Preliminaries and main results

Let us first introduce some notation. We denote by Ω_ℓ the open subset of $\mathbb{R}^n$ defined as

$$\Omega_\ell = (-\ell, \ell)^p \times \omega.$$

ℓ is a positive number, $1 \le p < n$ an integer, ω is a bounded open subset of $\mathbb{R}^{n-p}$. The points in $\mathbb{R}^n$ will be denoted by $x = (X_1, X_2)$, with

$$X_1 = (x_1, \dots, x_p), \qquad X_2 = (x_{p+1}, \dots, x_n).$$

We put

$$\nabla_1 = (\partial_{x_1}, \dots, \partial_{x_p}), \qquad \nabla_2 = (\partial_{x_{p+1}}, \dots, \partial_{x_n}).$$

Let $A = A(X_1, X_2)$ be a $n \times n$ symmetric matrix of the type

$$A = A(X_1, X_2) = \begin{pmatrix} A_{11}(X_1, X_2) & A_{12}(X_2) \\ A_{12}^T(X_2) & A_{22}(X_2) \end{pmatrix}. \tag{2.1}$$

In (2.1), A_{11} is a $p \times p$ symmetric matrix. We will assume that A is uniformly bounded and uniformly positive definite on $\mathbb{R}^p \times \omega$ that is to say that

$$|A(x)| \le \Lambda \qquad \text{a.e. } x \in \mathbb{R}^p \times \omega, \tag{2.2}$$

$$A(x)\xi \cdot \xi \ge \lambda |\xi|^2 \qquad \text{a.e. } x \in \mathbb{R}^p \times \omega, \quad \forall \xi \in \mathbb{R}^n, \tag{2.3}$$

where Λ and λ are positive constants. For

$$m \in L^\infty(\omega), \qquad m \ne 0, \qquad m \ge 0 \text{ in } \omega, \tag{2.4}$$

we would like to consider the eigenvalue problem

$$\begin{cases} u_\ell \in H_0^1(\Omega_\ell), u_\ell \ne 0, \qquad \lambda_\ell \in \mathbb{R}, \\ -\operatorname{div}\big(A(x)\nabla u_\ell\big) = \lambda_\ell m(X_2) u_\ell \qquad \text{in } H^{-1}(\Omega_\ell). \end{cases} \tag{2.5}$$

When $\ell \to +\infty$, we expect that the limit eigenvalue problem (2.5) will be the eigenvalue problem on ω defined as

$$\begin{cases} u \in H_0^1(\omega), u \neq 0, \qquad \mu \in \mathbb{R}, \\ -\operatorname{div}\big(A_{22}(X_2)\nabla_2 u\big) = \mu m(X_2) u \qquad \text{in } H^{-1}(\omega). \end{cases} \tag{2.6}$$

According to [1], Theorem 0.6, we have the

Theorem 2.1. *Under the above notation and assumptions, the problem* (2.5) *has a sequence*

$$0 < \lambda_\ell^1 < \lambda_\ell^2 \le \cdots \le \lambda_\ell^k \le \cdots$$

of eigenvalues. The first eigenvalue λ_ℓ^1 is simple and the corresponding eigenfunctions do not change sign in Ω_ℓ. We denote by u_ℓ^1 the positive eigenvalue such that $|u_\ell^1|_{\infty,\Omega_\ell} = 1$. We will also let u_ℓ^k be the eigenfunctions corresponding to λ_ℓ^k normalized by

$$|u_\ell^k|_{\infty,\Omega_\ell} = 1, \qquad \int_{\Omega_\ell} u_\ell^k u_\ell^h \,\mathrm{d}x = 0 \qquad h \neq k. \tag{2.7}$$

Moreover, the following variational characterization holds

$$\lambda_\ell^k = \operatorname{Inf}\left\{ \int_{\Omega_\ell} A\nabla u \cdot \nabla u \,\mathrm{d}x \;\mid u \in H_0^1(\Omega_\ell), \int_{\Omega_\ell} m u^2 \,\mathrm{d}x = 1, \right.$$
$$\left. \int_{\Omega_\ell} u u_\ell^i \,\mathrm{d}x = 0 \textit{ for all } i = 1, \dots, k-1 \right\}. \tag{2.8}$$

Of course the same kind of statement holds true for problem (2.6) and we denote by μ^1 the first eigenvalue of (2.6). Recall that the eigenfunctions of (2.5) and (2.6) belong to L^∞ according to [7], Theorem 8.15.

We state now our main result regarding the eigenvalues of (2.5).

Theorem 2.2. *Under the above notation and assumptions, it holds that*

$$\mu^1 \le \lambda_\ell^k \le \mu^1 + \frac{C_k}{\ell^2} \tag{2.9}$$

where $C_k = C(k, p, |A_{11}|_\infty)$, $|A_{11}|_\infty = \sup_{\mathbb{R}^n} |A_{11}(\cdot)|$, $|\cdot|$ is the norm of matrices subordinated to the Euclidean norm.

We notice that the convergence rate in (2.9) is optimal: see [3].

The next theorems deal with the eigenfunction of (2.5) when the matrix A has a diagonal structure and $p = 1$.

Theorem 2.3. *Under the assumptions of Theorem* 2.2*, assume in addition that $p = 1$ and $A_{12} = 0$, i.e.,*

$$A(x) = \begin{pmatrix} A_{11}(x) & 0 \\ 0 & A_{22}(X_2) \end{pmatrix}. \tag{2.10}$$

Then, for any first eigenfunction w of (2.6) *and $k \in \mathbb{N}$ fixed, a subsequence of $(u_\ell^k)_{\ell>0}$ converges in $H^1(\Omega_{\ell_0})$ as ℓ goes to infinity toward αw, for some $\alpha \in \mathbb{R}$. Moreover, if $\alpha \neq 0$ then a subsequence of $(\frac{1}{\alpha} u_\ell^k)_{\ell>0}$ converges in $H^1(\Omega_{\ell_0})$ toward w.*

Our last result concerns classical eigenvalue problems corresponding to the case $m = 1$. We refer to [3] for other results in this direction.

Theorem 2.4. *Under the assumptions of Theorem* 2.3, *assume in addition that* $m = 1$ *and* $A_{11}(x) = A_{11}(X_1) = A_{11}(-X_1)$ *a.e.* $x \in \mathbb{R}^n$, $X_1 \in \mathbb{R}$. *Let* w *be the first positive eigenfunction of* (2.6) *with* $|w|_{\infty,\omega} = 1$. *Then, for every positive* ℓ_0 *fixed,* u_ℓ^1 *converges toward* w *in* $H^1(\Omega_{\ell_0})$.

3. Proofs

3.1. Convergence of the eigenvalues

We start with the sketch of the proof of Theorem 2.2 which is similar to the proof of Theorem 2.1 in [3]. In particular, an easy modification of [3], Lemma 2.2, gives the

Lemma 3.1. *Let* Ω *be a bounded open set of* $\mathbb{R}^n$. *Let* A_ε *be a family of symmetric matrices such that for* λ, Λ *positive independent of* ε *we have*

$$\begin{aligned} &\lambda|\xi|^2 \le A_\varepsilon(x)\xi\cdot\xi && \text{a.e. } x\in\Omega, \quad \forall\,\xi\in\mathbb{R}^n,\\ &|A(x)| \le \Lambda && \text{a.e. } x\in\Omega. \end{aligned}$$

Suppose that

$$A_\varepsilon(x) \to A(x) \quad \text{a.e. } x \in \Omega \text{ as } \varepsilon \to 0.$$

For $m \in L^\infty(\Omega)\setminus\{0\}$, $m \ge 0$ *a.e. in* Ω, *set*

$$\lambda_\varepsilon = \operatorname{Inf}\left\{\int_\Omega A_\varepsilon\nabla u\cdot\nabla u\,\mathrm{d}x \mid u\in H_0^1(\Omega),\ \int_\Omega mu^2\,\mathrm{d}x = 1\right\}, \tag{3.1}$$

$$\lambda_0 = \operatorname{Inf}\left\{\int_\Omega A\nabla u\cdot\nabla u\,\mathrm{d}x \mid u\in H_0^1(\Omega),\ \int_\Omega mu^2\,\mathrm{d}x = 1\right\}, \tag{3.2}$$

then we have

$$\lim_{\varepsilon\to0}\lambda_\varepsilon = \lambda_0, \qquad \lim_{\varepsilon\to0} u_\varepsilon = u_0 \qquad \text{in } H_0^1(\Omega),$$

where u_ε, u_0 *denote respectively the first positive eigenfunctions realizing the infimum of* (3.1), (3.2).

We are now in the position to give the sketch of the

Proof of Theorem 2.2. Let us first establish the lower bound in (2.9). For this consider A_ε defined by

$$A_\varepsilon = \begin{cases} A & \text{on } \Omega_{\ell-\varepsilon},\\[2mm] \left(\begin{array}{ccc|c} \lambda & & 0 & \\ & \ddots & & 0\\ 0 & & \lambda & \\ \hline & 0 & & A_{22} \end{array}\right) & \text{on } \Omega_\ell\setminus\Omega_{\ell-\varepsilon}, \end{cases}$$

where λ is the constant in the inequality (2.3). Clearly A_ε satisfies all the assumptions of Lemma 3.1 with $\Omega = \Omega_\ell$. Define

$$\lambda^1_{\ell,\varepsilon} = \operatorname{Inf}\left\{ \int_{\Omega_\ell} A_\varepsilon \nabla u \cdot \nabla u \,\mathrm{d}x \mid u \in H^1_0(\Omega_\ell),\ \int_{\Omega_\ell} m u^2 \,\mathrm{d}x = 1 \right\}, \tag{3.3}$$

the first generalized eigenvalue of the operator $-\operatorname{div}(A_\varepsilon \nabla \cdot)$ with Dirichlet boundary conditions. Denote also by u_ε the function – that we can assume > 0 – where the infimum of (3.3) is achieved. We have

$$\int_{\Omega_\ell} A_\varepsilon(x) \nabla u_\varepsilon \cdot \nabla v \,\mathrm{d}x = \lambda^1_{\ell,\varepsilon} \int_{\Omega_\ell} m u_\varepsilon v \,\mathrm{d}x \quad \forall\, v \in H^1_0(\Omega_\ell). \tag{3.4}$$

Denote then by ϱ_ε a smooth function on $[-\ell, \ell]$ such that

$$0 \le \varrho_\varepsilon \le 1, \quad \varrho_\varepsilon = 1 \text{ on } (-\ell + \varepsilon, \ell - \varepsilon), \quad \varrho_\varepsilon(\pm \ell) = 0, \quad \varrho_\varepsilon \text{ is concave.} \tag{3.5}$$

Let w be a first positive eigenfunction to (2.6) and in (3.4) choose

$$v = w(X_2) \prod_{i=1}^{p} \varrho_\varepsilon(x_i).$$

We obtain (we decompose A_ε as A in (2.1) and put the ε above as an upper index)

$$\int_{\Omega_\ell} \{ A^\varepsilon_{11} \nabla_1 u_\varepsilon \cdot \nabla_1 v + A^\varepsilon_{12} \nabla_2 u_\varepsilon \cdot \nabla_1 v + A^{\varepsilon T}_{12} \nabla_1 u_\varepsilon \cdot \nabla_2 v + A^\varepsilon_{22} \nabla_2 u_\varepsilon \cdot \nabla_2 v \} \,\mathrm{d}x$$
$$= \lambda^1_{\ell,\varepsilon} \int_{\Omega_\ell} m u_\varepsilon v \,\mathrm{d}x.$$

Arguing as in the proof of [3] Theorem 2.1, we arrive at

$$\mu^1 \int_{\Omega_\ell} m u_0 w \,\mathrm{d}x \le \lambda^1_\ell \int_{\Omega_\ell} m u_0 w \,\mathrm{d}x$$

which implies the first inequality of (2.9) since u_0, w are positive and $m \ge 0$, $m \ne 0$. We will now prove the second inequality of (2.9). In the particular case where $k = 1$, we can show (with a proof like in [3]) that

$$\lambda^1_\ell \le \mu^1 + \frac{p\pi^2 |A_{11}|_\infty}{4\ell^2}.$$

When $k \ge 2$, we split the domain Ω_ℓ into k subdomains in the x_1-direction – i.e., we set

$$Q_i = \left(-\ell + (i-1)\frac{2\ell}{k}, -\ell + i\frac{2\ell}{k} \right) \times (-\ell, \ell)^{p-1} \times \omega \qquad i = 1, \dots, k. \tag{3.6}$$

Moreover, applying Theorem 2.1 (we notice that by (2.4) and (3.6), $m \ge 0$, $m \ne 0$ in Q_i), we denote by $\lambda^1_{Q_i}$ the first eigenvalue defined by

$$\lambda^1_{Q_i} = \lambda^1_{Q_i}(A) = \operatorname{Inf}\left\{ \int_{Q_i} A \nabla u \cdot \nabla u \,\mathrm{d}x \mid u \in H^1_0(Q_i),\ \int_{Q_i} m u^2 \,\mathrm{d}x = 1 \right\}$$

and by u_i the first eigenfunction – i.e., the only positive function achieving the infimum above. Arguing as in [3], we then obtain

$$\lambda_\ell^k \le \mu^1 + \frac{pk^2\pi^2|A_{11}|_\infty}{4\ell^2}$$

which completes the proof of the theorem. □

3.2. Convergence of the eigenfunctions

The proof of Theorem 2.3 will use the two lemmas below. Without loss of generality, we may assume that $\int_\omega w^2 \,\mathrm{d}X_2 = 1$. In order to specify this normalization, we will denote w by w_1.

Lemma 3.2. *Under the assumptions of Theorem* 2.3*, we have*

$$\begin{aligned} &\big||\nabla_1 u_\ell^k|\big|_{2,\Omega_\ell} = O\big(\frac{1}{\ell^{1/2}}\big),\\ &u_\ell^k - \big(\int_\omega u_\ell^k w_1 \,\mathrm{d}X_2\big) w_1 \to 0 \qquad \textit{in } H_0^1(\Omega_\ell). \end{aligned} \tag{3.7}$$

Proof. Let us decompose u_ℓ^k under the form

$$u_\ell^k = \big(\int_\omega u_\ell^k w_1 \,\mathrm{d}X_2\big) w_1 + r_\ell = \alpha w_1 + r. \tag{3.8}$$

Let μ^2 be the second eigenvalue of (2.6). We remark that, for a.e. $X_1 \in (-\ell,\ell)$, w_1 and $r(X_1,\cdot)$ are orthogonal in $L^2(\omega)$, hence for $\varepsilon > 0$ satisfying $(1-\varepsilon)\mu^2 = \mu^1$, we have with the characterization (2.8)

$$\begin{aligned} \int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2 &= (1-\varepsilon)\int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2 + \varepsilon\int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2\\ &\ge (1-\varepsilon)\mu^2\int_\omega m r^2 \,\mathrm{d}X_2 + \varepsilon\int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2\\ &= \mu^1\int_\omega m r^2 \,\mathrm{d}X_2 + \varepsilon\int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2. \end{aligned} \tag{3.9}$$

Thus, using (3.8), (3.9) and (2.6), $\int_\omega A_{22}\nabla_2 u_\ell^k \nabla_2 u_\ell^k \,\mathrm{d}X_2$ is greater or equal to

$$\begin{aligned} &\alpha^2\int_\omega A_{22}\nabla_2 w_1\cdot\nabla_2 w_1 \,\mathrm{d}X_2 + \int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2\\ &\quad + 2\alpha\int_\omega A_{22}\nabla_2 w_1\cdot\nabla_2 r \,\mathrm{d}X_2\\ &\ge \mu^1\int_\omega m(\alpha^2 w_1^2 + r^2) \,\mathrm{d}X_2 + \varepsilon\int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2\\ &\quad + 2\alpha\mu^1\int_\omega m w_1 r \,\mathrm{d}X_2\\ &= \mu^1\int_\omega m(u_\ell^k)^2 \,\mathrm{d}X_2 + \varepsilon\int_\omega A_{22}\nabla_2 r\cdot\nabla_2 r \,\mathrm{d}X_2. \end{aligned} \tag{3.10}$$

Testing (2.5) with u_ℓ^k, we obtain thanks to (2.10), (2.3) and (3.10),

$$\lambda\int_{\Omega_\ell}|\nabla_1 u_\ell^k|^2\,\mathrm{d}x+\lambda\varepsilon\int_{\Omega_\ell}|\nabla_2 r|^2\,\mathrm{d}x\le(\lambda_\ell^k-\mu^1)\int_{\Omega_\ell}m(u_\ell^k)^2\,\mathrm{d}x.$$

Then

$$\big|\nabla_1 u_\ell^k\big|_{2,\Omega_\ell}+\big|\nabla_2 r\big|_{2,\Omega_\ell}=O\big(\frac{1}{\ell^{1/2}}\big),$$

since $\lambda_\ell^k-\mu^1=O\big(\frac{1}{\ell^2}\big)$ by Theorem 2.2, $m\in L^\infty(\omega)$ and $|u_\ell^k|_{\infty,\Omega_\ell}=1$. To control the L^2 norm of r, we use the following "anisotropic" Poincaré inequality: there exists a constant $C(\omega)$ depending only on ω such that, for every $v\in H_0^1(\Omega_\ell)$, $|v|_{2,\Omega_\ell}\le C(\omega)\big|\nabla_2 v\big|_{2,\Omega_\ell}$ (see [2] or [4] for a proof). Next by (3.8) and the Cauchy-Schwarz inequality,

$$\big|\nabla_1 r\big|_{2,\Omega_\ell}\le\big|\nabla_1 u_\ell^k\big|_{2,\Omega_\ell}+\big|\nabla_1(\textstyle\int_\omega u_\ell^k w_1\,\mathrm{d}X_2)w_1\big|_{2,\Omega_\ell}\le 2\big|\nabla_1 u_\ell^k\big|_{2,\Omega_\ell}.$$

Therefore $\big|\nabla_1 r\big|_{2,\Omega_\ell}=O\big(\frac{1}{\ell^{1/2}}\big)$ which completes the proof of the lemma. □

The convergence (3.7) expresses that, in some sense, u_ℓ^k is closed to a function with separated variables. We will use this property in the sequel to get local strong convergence of a subsequence of u_ℓ^k.

Lemma 3.3. *For all $\ell_0>0$, there exists a constant $C(\ell_0)$ such that*

$$\big|\nabla u_\ell^k\big|_{2,\Omega_{\ell_0}}\le C(\ell_0)\qquad\forall\ell>0.$$

Proof. Since $\big|\nabla_1 u_\ell^k\big|_{2,\Omega_{\ell_0}}$ is bounded by Lemma 3.2, it remains to estimate $\big|\nabla_2 u_\ell^k\big|_{2,\Omega_{\ell_0}}$. The function $U_\ell=(\int_\omega u_\ell^k w_1\,\mathrm{d}X_2)w_1$ satisfies, since $|u_\ell^k|_{\infty,\Omega_\ell}=1$,

$$\begin{aligned}\big|\nabla_2 U_\ell\big|_{2,\Omega_{\ell_0}}^2&=\int_{\Omega_{\ell_0}}\big|(\int_\omega u_\ell^k w_1\,\mathrm{d}X_2)\nabla_2 w_1\big|^2\,\mathrm{d}x\\&\le\int_{\Omega_{\ell_0}}(\int_\omega w_1\,\mathrm{d}X_2)^2|\nabla_2 w_1|^2\,\mathrm{d}x\le C(\ell_0).\end{aligned}$$

Hence by Lemma 3.2. (see also (3.8))

$$\big|\nabla_2 u_\ell^k\big|_{2,\Omega_{\ell_0}}\le\big|\nabla_2 U_\ell\big|_{2,\Omega_{\ell_0}}+\big|\nabla_2 r\big|_{2,\Omega_{\ell_0}}\le C(\ell_0).$$ □

Proof of Theorem 2.3. The following convergences are understood up to a subsequence. The sequence $(u_\ell^k)_{\ell>0}$ is bounded in $H^1(\Omega_{\ell_0})$ by Lemma 3.3 and $|u_\ell^k|_{\infty,\Omega_\ell}=1$, therefore $u_\ell^k\rightharpoonup u_0$ in $H^1(\Omega_{\ell_0})$ and

$$u_\ell^k\to u_0\qquad\text{in } L^2(\Omega_{\ell_0}),\tag{3.11}$$

$$\nabla_1 u_\ell^k\rightharpoonup\nabla_1 u_0\qquad\text{in } L^2(\Omega_{\ell_0}).\tag{3.12}$$

But $\nabla_1 u_\ell^k\to 0$ in $L^2(\Omega_{\ell_0})$ by Lemma 3.2. Thus $u_0=u_0(X_2)$ is independent of X_1.

Now we claim that

$$u_0=(\int_\omega u_0 w_1\,\mathrm{d}X_2)w_1=\alpha_1 w_1.\tag{3.13}$$

Indeed, with the Cauchy-Schwarz inequality, we get since $|w_1|_{2,\omega} = 1$,

$$|(\int_\omega (u_\ell^k - u_0) w_1 \,\mathrm{d}X_2) w_1|^2_{2,\Omega_{\ell_0}} \leq \int_{\Omega_{\ell_0}} (u_\ell^k - u_0)^2 \,\mathrm{d}x \xrightarrow[\ell\to\infty]{} 0,$$

by (3.11). Thus

$$(\int_\omega u_\ell^k w_1 \,\mathrm{d}X_2) w_1 \to (\int_\omega u_0 w_1 \,\mathrm{d}X_2) w_1 \qquad \text{in } L^2(\Omega_{\ell_0})$$

and (3.13) follows from (3.7). By (3.13) and Lemma 3.2,

$$\nabla_1(u_\ell^k - u_0) = \nabla_1 u_\ell^k \to 0 \qquad \text{in } L^2(\Omega_{\ell_0}),$$

thus since $u_\ell^k \to u_0$ in $L^2(\Omega_{\ell_0})$, it remains to prove that

$$\nabla_2(u_\ell^k - u_0) \to 0 \qquad \text{in } L^2(\Omega_{\ell_0}). \tag{3.14}$$

Recalling (3.13), we have

$$\int_{\Omega_{\ell_0}} \left| \nabla_2 \left\{ \int_\omega u_\ell^k w_1 \,\mathrm{d}X_2\, w_1 - \alpha_1 w_1 \right\} \right|^2 \mathrm{d}x = \int_{\Omega_{\ell_0}} \left| \int_\omega (u_\ell^k - u_0) w_1 \,\mathrm{d}X_2 \nabla_2 w_1 \right|^2 \mathrm{d}x.$$

According to (2.3), (2.4), this latter integral is bounded by

$$\frac{\mu^1}{\lambda} |m|_{\infty,\omega} \int_{-\ell_0}^{\ell_0} \left(\int_\omega (u_\ell^k - u_0) w_1 \,\mathrm{d}X_2 \right)^2 \mathrm{d}X_1.$$

Now, by the Cauchy-Schwarz inequality, it is less or equal to

$$C \int_{\Omega_{\ell_0}} (u_\ell^k - u_0)^2 \,\mathrm{d}x \xrightarrow[\ell\to\infty]{} 0,$$

by (3.11). We have proved that

$$\nabla_2 (\int_\omega u_\ell^k w_1 \,\mathrm{d}X_2) w_1 \to \nabla_2 \alpha_1 w_1 \qquad \text{in } L^2(\Omega_{\ell_0}).$$

We then show (3.14) using (3.7). □

Remark 3.4. If A is the identity matrix of $\mathbb{R}^n$, $m = 1$ a.e. in Ω and $k = 2$ (that is, if we consider the sequence of the second eigenfunctions of the Laplace operator with homogeneous Dirichlet boundary conditions) then $\alpha_1 = 0$.

In the proof of Theorem 2.4, we will deal with eigenfunctions in separated variables form in the case where $p = 1$. Considering first the one dimensional problem

$$\begin{cases} u_\ell \in H_0^1\big((-\ell,\ell)\big), u_\ell \neq 0, \qquad \lambda_\ell \in \mathbb{R}, \\ -\big(a(x) u_\ell'\big)' = \lambda_\ell u_\ell \quad \text{in } H^{-1}\big((-\ell,\ell)\big), \end{cases} \tag{3.15}$$

we make, besides the analogue of (2.2) and (2.3), the following structural assumption on the function $a \in L^\infty(\mathbb{R})$:

$$a(x) = a(-x) \qquad \text{a.e } x \in \mathbb{R}. \tag{3.16}$$

We have the

Lemma 3.5. *Let u_ℓ be the first positive eigenfunction of* (3.15)*. If* (3.16) *holds then u_ℓ is continuous in $[-\ell,\ell]$ and*

$$u_\ell(0) = |u_\ell|_{\infty,(-\ell,\ell)}. \tag{3.17}$$

Proof. By (3.16), $x \mapsto u_\ell(-x)$ is also solution to (3.15) thus

$$u_\ell(-x) = u_\ell(x) \quad \forall x \in (-\ell,\ell). \tag{3.18}$$

Since u_ℓ is positive in $(-\ell,\ell)$, we deduce from (3.15) that $a(\cdot)u_\ell'$ is non-increasing in $(-\ell,\ell)$. Moreover, u_ℓ is continuously differentiable hence, by (3.18), $u_\ell'(0) = 0$. Thus 0 is the maximizer of u_ℓ since a is positive and (3.17) follows. □

Lemma 3.6. *Under the assumption of Lemma* 3.5*, assume in addition that*

$$|u_\ell|_{\infty,(-\ell,\ell)} = 1.$$

Then, for every positive ℓ_0 fixed,

$$u_\ell \to 1 \quad \textit{in } H^1\big((-\ell_0,\ell_0)\big).$$

Proof. Since $|u_\ell|_{\infty,(-\ell,\ell)} = 1$, and $\lambda_\ell = O(\frac{1}{\ell^2})$,

$$\lambda \int_{-\ell_0}^{\ell_0} (u_\ell')^2 \,\mathrm{d}x \le \int_{-\ell}^{\ell} a(x)(u_\ell')^2 \,\mathrm{d}x = \lambda_\ell \int_{-\ell}^{\ell} u_\ell^2 \,\mathrm{d}x \le 2\ell\lambda_\ell \xrightarrow[\ell\to\infty]{} 0.$$

Hence, up to a subsequence, there exists $u_0 \in \mathbb{R}$ such that

$$u_\ell \to u_0 \quad \text{in } H^1\big((-\ell_0,\ell_0)\big) \text{ and in } C\big([-\ell_0,\ell_0]\big).$$

Thus $u_0 = 1$ by Lemma 3.5 and the whole sequence u_ℓ convergences by uniqueness of the limit. □

Proof of Theorem 2.4. Since $m = 1$, we know that $u_\ell^1(X_1,X_2) = v_\ell(X_1)w(X_2)$ where v_ℓ and w are the first positive eigenfunctions of (3.15) and (2.6) with $|v_\ell|_{\infty,(-\ell,\ell)} = 1$ and $|w|_{\infty,\omega} = 1$ respectively. Hence by Lemma 3.6,

$$u_\ell^1 - w = (v_\ell - 1)w \to 0 \quad \text{in } H^1(\Omega_{\ell_0}).$$ □

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Hyperbolic Problems and Regularity Questions
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Loss of Derivatives for $t \to \infty$ in Strictly Hyperbolic Cauchy Problems

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Abstract. We study the behavior for $t \to \infty$ of the solutions to the Cauchy problem for a strictly hyperbolic second order equation with coefficients periodic in time, or oscillating with a period going to 0.

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1. Introduction

Let us consider the Cauchy problem in $[0, +\infty) \times \mathbf{R}^n_x$

$$\begin{cases} \partial_t^2 u - \displaystyle\sum_{i,j=1}^{n} a_{ij}(t) \partial^2_{x_i x_j} u = 0 \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.1}$$

with data $u_0 \in H^s(\mathbf{R}^n)$, $u_1 \in H^{s-1}(\mathbf{R}^n)$, $s > 0$, under the strict hyperbolicity assumption

$$0 < \lambda \leq \sum_{i,j=1}^{n} a_{ij}(t)\xi_i \xi_j / |\xi|^2 \leq \Lambda \quad \text{for all } \xi \in \mathbf{R}^n \setminus \{0\}. \tag{1.2}$$

For simplicity's sake, we will consider the model problem in one space dimension:

$$\begin{cases} \partial_t^2 u - a(t) \partial_x^2 u = 0, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases} \tag{1.3}$$

but there is no difficulty to extend our arguments to the case $x \in \mathbf{R}^n$, $n \geq 2$.

It is well known that if the coefficients $a_{ij}(t)$ are Lipschitz continuous, then the Cauchy problem (1.1) is C^∞ well posed. More precisely, in this case we have

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well-posedness in Sobolev spaces and for any given Cauchy data $u_0 \in H^s(\mathbf{R})$, $u_1 \in H^{s-1}(\mathbf{R})$ there is a unique solution

$$u \in C\left([0,+\infty); H^s(\mathbf{R})\right) \cap C^1\left([0,+\infty); H^{s-1}(\mathbf{R})\right).$$

Moreover, denoting

$$E_s(u)(t) := \|u(t)\|^2_{H^s} + \|\partial_t u(t)\|^2_{H^{s-1}}\,, \tag{1.4}$$

the solution u, for any $T > 0$, any $s \in \mathbf{R}$ and any $t \in [0,T]$, satisfies the estimate

$$E_s(u)(t) \le C_{s,T} E_s(u)(0)\,, \tag{1.5}$$

with $C_{s,T} > 0$ (see, e.g., [11], Chapter 9 or [12], Chapter 6).

One can pose different problems, related to (1.1), seemingly very simple. One can ask, e.g., if in order to have C^∞ well-posedness for (1.1) the Lipschitz continuity of the coefficients $a_{ij}(t)$ is necessary: indeed the Lipschitz continuity can be substituted by the so-called Log-Lip property. We recall the definition:

Definition 1.1. A function $f : I \to \mathbf{R}$, I a real interval, is said Log-Lip continuous if it satisfies

$$\|f\|_{LL(I)} := \sup_{\substack{t,t+\tau\in I\\ 0<|\tau|<1/2}} \frac{|f(t+\tau) - f(t)|}{|\tau|\,|\log|\tau||} < +\infty. \tag{1.6}$$

The sufficiency of the Log-Lip regularity of the coefficients in order to obtain the C^∞ well-posedness was first proved in [3] for the case of coefficients independent of x variables, by using the method of the so-called approximate energies, firstly introduced in [3]. After this, the case of an equation with coefficients Log-Lipschitz continuous depending on all the variables have been treated in [7] for an equation of the form as in (1.1), and finally in [8] for the general case of a second order hyperbolic equation; moreover in [8] it is proved that for such equations with Log-Lip coefficients the C^∞ local uniqueness property is verified. In [7] and in [8] the well-posedness was obtained by using the method of the approximate energies coupled with paradifferential calculus (see [1]) suitably extended to Log-Lipschitz continuous functions.

One can also remark that the Log-Lip regularity is the minimal regularity assumption on the coefficients in order to have the C^∞ well-posedness: in [3], by constructing a coefficient $a(t)$ which is Hölder continuous of any exponent smaller than 1, an example shows that the Log-Lip assumption cannot be weakened greatly. In [7], generalizing such example, it is proved that under any weaker hypothesis than the Log-Lip regularity, the Cauchy problem (1.1) is not in general well-posed in C^∞. Moreover, in [7] it is proved that an energy estimate with a loss of derivatives is satisfied in general. More precisely the solution u, for any $T > 0$, any $s \in \mathbf{R}$ and any $t \in [0,T]$, satisfies the estimate:

$$E_{s-\beta t}(u)(t) \le C^*_{s,T} E_s(u)(0), \tag{1.7}$$

where $C^*_{s,T}$ is a positive constant depending only on s, T, the dimension n and Λ (see (1.2)), and where the constant β is given by

$$\beta = \frac{1}{\lambda} C^* \|a\|_{LL([0,T])} \tag{1.8}$$

with C* a positive constant depending only on n and λ the bound from below in (1.2).

In [2], answering to some open problems posed in [13] and [14], the authors show, by examples, that in general a loss really occurs for any *slightly worse* regularity than Lip and that in the Log-Lip case the loss of derivatives cannot be arbitrarily small.

Strictly related to these questions is the study of the behavior of the solution to the hyperbolic problem (1.1) when $t \to +\infty$. One can consider the case of Lipschitz continuous, or Log-Lip continuous, or even slightly less regular coefficients. One can ask in particular what happens for a coefficient $a(t)$ periodic or, more generally, such that, for a sequence of intervals going from 0 to $+\infty$, it is periodic in each interval of the sequence, with period or amplitude constant, or going to 0 for $t \to +\infty$.

We will show that in each of these cases the solution of (1.1) may blow up, in the Sobolev norms, or even in the ultradistributions spaces $(\mathcal{D}^\sigma)'$. More precisely, we will give 3 types of examples: in the first one the coefficient $a(t)$ will be globally periodic in $[0, +\infty)$; in the second one $a(t)$ is periodic in each interval $[k, k+1)$: the period goes to 0, for $k \to +\infty$, while the amplitude remains constant; finally, in the third one, $a(t)$ is again periodic in each interval $[k, k+1)$, but now, for $k \to +\infty$, the period and also the amplitude go to 0; moreover $a(t)$ is "almost" Log-Lip continuous on $[0, +\infty)$. In all these cases, we give 2 initial data in H^s for every $s \in \mathbf{R}$ such that the corresponding solution $u(t, x)$ blows up, when $t \to +\infty$, at least in H^s for any real number s.

Moreover, we will consider the case of 2 sequences of intervals going to $+\infty$, and a coefficient $a(t)$ which is constant in any interval of one sequence, periodic in the intervals of the other. For such coefficients, we will give 3 examples, corresponding to 3 cases above stated.

2. Main results

Let us consider the Cauchy problem (1.3) under the condition of strict hyperbolicity (1.2). In the following theorems, we consider 2π-periodic Cauchy data and solutions. In order to simplify the proofs, we will consider the Cauchy problem for $x \in \mathbf{T}$ instead of $x \in \mathbf{R}$, where $\mathbf{T}$ denotes the one dimensional torus $\mathbf{T} = \mathbf{R}/2\pi\mathbf{Z}$. Moreover, instead of the energy $E_s(u)(t)$ in (1.4), here we use

$$\dot{E}_s(u)(t) = \|u(t)\|_{\dot{H}^s} + \|\partial_t u(t)\|_{\dot{H}^{s-1}} \tag{2.1}$$

where $\dot{H}^s$ denotes the homogeneous Sobolev space of exponent s on the one dimensional torus $\mathbf{T}$.

Before we state the Theorem, we need a definition.

Definition 2.1. A function $\Omega \in C^1((0,\delta])$ for some $\delta > 0$ (we can always assume $\delta < 1/2$), is said to be a *modulus function* if it is a convex, positive, decreasing function such that

$$\lim_{\tau\to 0^+} \Omega(\tau) = +\infty\,, \quad 0 < -\,\Omega'(\tau) \le \tau^{-1}, \quad \Omega(\delta) \ge 1\,. \tag{2.2}$$

We remark that Definition 2.1. is satisfied, e.g., by the function $\Omega(\tau) = \log(\tau)$ or, more generally, by $\Omega(\tau) = \log^{(p)}(\tau)$, where $\log^{(1)}(\tau) = \log(\tau)$ and, for $p > 1$, $\log^{(p)}(\tau) = \log\big(\log^{(p-1)}(\tau)\big)$.

In relation to a function Ω verifying Definition 2.1. we give the following:

Definition 2.2. A function $f : I \to \mathbf{R}$, I a real interval, is said Ω-Log-Lip continuous ($f \in \Omega LL(I)$) if it satisfies

$$\|f\|_{\Omega LL(I)} := \sup_{\substack{t,t+\tau\in I\\ 0<|\tau|<\delta}} \frac{|f(t+\tau)-f(t)|}{|\tau|\,|\log|\tau||\,\Omega(|\tau|)} < +\infty.$$

We remark that, for $\Omega(\tau) = |\log|\tau||^{-1}\,\tau^{-\alpha}$, $\Omega LL(I)$ coincides with the usual Hölder space $C^{0,1-\alpha}(I)$, while, formally, if we take $\Omega(\tau) = 1$, $\Omega LL(I)$ would coincide with the Log-Lip class.

Moreover, we denote by $\gamma^\sigma = \gamma^\sigma(\mathbf{T})$ the space of the Gevrey functions of index σ $\mathbf{T}$-periodic and by $(\mathcal{D}^\sigma)' = (\mathcal{D}^\sigma(\mathbf{T}))'$ the corresponding space of the Gevrey ultradistributions of index σ, $\sigma > 1$.

On the other hand, let us set:

$$I_k = [k, k+1)\,, \quad k = 0, 1, \dots \tag{2.3}$$

We will always have:

$$a \in C^\infty([0,+\infty))\,. \tag{2.4}$$

We have then the following Theorem:

Theorem 2.3.

i) *There are an analytic function $a(t)$ verifying* (1.2)*, periodic in $\mathbf{R}$, and two Cauchy data $u_0, u_1 \in H^s(\mathbf{T}) \cap \gamma^\sigma(\mathbf{T})$ for every $s \in \mathbf{R}$ and every $\sigma > 1$, such that the solution u of the Cauchy problem* (1.3) *verifies:*

$$\textit{for any } s \in \mathbf{R} \quad \lim_{t\to+\infty} \dot{E}_s(u)(t) = +\infty, \tag{2.5}$$

$$\textit{for any } \sigma > 1 \textit{ and any } \bar{t}\,,\ u(t,\cdot) \textit{ is unbounded in } (\mathcal{D}^\sigma)' \textit{ for } t \in (\bar{t}, +\infty). \tag{2.6}$$

ii) *There are a function $a(t)$ verifying* (1.2) *and* (2.4)*, periodic in any interval I_k with a period P_k, verifying, with $0 < l_1 < 1 < l_2$,*

$$\lim_{k\to+\infty} P_k = 0\,; \quad \textit{for every } k \in \mathbf{N} \quad \min_{t\in I_k} a(t) = l_1\,, \quad \max_{t\in I_k} a(t) = l_2 \tag{2.7}$$

and two Cauchy data $u_0, u_1 \in H^s(\mathbf{T}) \cap \gamma^\sigma(\mathbf{T})$ *for every* $s \in \mathbf{R}$ *and every* $\sigma > 1$*, such that the solution* u *of the Cauchy problem* (1.3) *verifies*

$$\textit{for any } s \in \mathbf{R} \quad \limsup_{t \to +\infty} \dot{E}_s(u)(t) = +\infty \tag{2.8}$$

and again (2.6).

iii) *For any modulus function* Ω *verifying* (2.2) *there are a function* $a \in \Omega LL([0, +\infty))$ *verifying* (1.2) *and* (2.4)*, periodic in any interval* I_k *with a period* P_k*, verifying*

$$\lim_{k \to +\infty} P_k = 0 \quad \textit{and} \quad \lim_{t \to +\infty} a(t) = 1\,, \tag{2.9}$$

and two Cauchy data $u_0, u_1 \in H^s(\mathbf{T})$ *for every* $s \in \mathbf{R}$*, such that the solution* u *of the Cauchy problem* (1.1) *verifies* (2.5).

Remark 2.4. We want to remark that in Theorem 2.2. part ii) we prove the blow up of the solution u of the Cauchy problem (1.3) in the sense that $\limsup_{t \to +\infty} \dot{E}_s(u)(t) = +\infty$ while in part iii) we obtain, more precisely, that $\lim_{t \to +\infty} \dot{E}_s(u)(t) = +\infty$.

Remark 2.5. The construction of an example with coefficient constant in a sequence of intervals, and periodic in another is completely analogous to the previous one. Instead of (2.3), we give the following definition:

$$I_k = [2k, 2k+1)\,, \quad J_k = [2k+1, 2k+2)\,, \quad k = 0, 1, \ldots \tag{2.10}$$

In all the cases we will define:

$$a(t) = 1 \quad \text{for} \quad t \in J_k\,, \tag{2.11}$$

while in the intervals I_k we give a definition analogous to that of Theorem 2.2. i)–iii). Condition (2.4) will be again satisfied in any case (while, obviously, we renounce to the analyticity of $a(t)$ in case i)), and all the conclusions are still valid.

Remark 2.6. In Theorem 2.3 i)–ii) we obtain a blow up phenomenon in any reasonable sense; in iii), where the period, and also the amplitude, of the coefficient $a(t)$ go to 0 for $t \to +\infty$, we are forced to renounce to the explosion in ultradistributions spaces $(\mathcal{D}^\sigma)'$; moreover, in order to obtain the explosion of the solution for $t \to +\infty$, at least in the spaces H^s (for every $s \in \mathbf{R}$), we are forced to renounce to Log-Lip regularity of the coefficient $a(t)$, by introducing the classes ΩLL.

In relation to these facts one could pose 2 questions:

1) For a coefficient $a(t)$ Log-Lip, or also Lipschitz continuous, verifying

$$\lim_{t \to +\infty} a(t) = 1, \tag{2.12}$$

is it possible to obtain a H^s energy estimate uniform for $t \in [0, +\infty)$?

2) For a coefficient $a(t)$ as above, again verifying (2.12), but not necessarily Log-Lipschitz continuous, is it possible to obtain an energy estimate uniform for $t \in [0, +\infty)$ in some Gevrey class of functions or ultradistributions?

We do not know the answer to these questions.

3. Proofs

Proof of Theorem 2.3. i) Let us define

$$a(t) = \alpha_{\overline{\varepsilon}}(t),$$

where (see [9] and [3]) we have set

$$\alpha_\varepsilon(\tau) = 1 - 4\varepsilon \sin(2\tau) - \varepsilon^2(1 - \cos(2\tau))^2 \tag{3.1}$$

and we have fixed $\overline{\varepsilon} = 1/10$ so that (1.2) is verified with $\lambda = 1/2$ and $\Lambda = 3/2$.

Let $u(t,x)$ be the solution of the Cauchy problem

$$\begin{cases} \partial_t^2 u - a(t)\partial_x^2 u = 0 \\ u(0,x) = 0,\ \partial_t u(0,x) = \sin(x). \end{cases} \tag{3.2}$$

Then, we have

$$u(t,x) = v(t)\sin(x),$$

where $v(t)$ is solution of

$$\begin{cases} v''(t) + a(t)v(t) = 0 \\ v(0) = 0,\ v'(0) = 1\,. \end{cases}$$

An easy calculation shows that

$$v(t) = \sin t \cdot \exp\left[\overline{\varepsilon}\left(t - \frac{1}{2}\sin 2t\right)\right]$$

and so, for $t \in [0, +\infty)$, we have:

$$v^2(t) + v'^2(t) \geq \frac{7}{10}\exp\left[2\,\overline{\varepsilon}\left(t - \frac{1}{2}\sin 2t\right)\right] \geq \frac{1}{2}\exp\left(\frac{1}{5}t\right)\,. \tag{3.3}$$

From (3.3), (2.5) and (2.6) follow immediately.

ii), iii) Let us take (see [10] and [4]) a real, non-negative, 2π-periodic, C^∞ function φ such that $\varphi(\tau) = 0$ for τ in a neighborhood of $\tau = 0$ and

$$\int_0^{2\pi} \varphi(\tau)\cos^2\tau\, d\tau = \pi.$$

Then, for every $\tau \in \mathbf{R}$ and $\varepsilon \in (0, \tilde{\varepsilon}]$ we define

$$\beta_\varepsilon(\tau) = 1 + 4\varepsilon\varphi(\tau)\sin 2\tau - 2\varepsilon\varphi'(\tau)\cos^2\tau - 4\varepsilon^2\varphi^2(\tau)\cos^4\tau\,, \tag{3.4}$$

where $\tilde{\varepsilon}$ will be chosen such that for $0 < \varepsilon \leq \tilde{\varepsilon}$ one has:

$$1/2 \leq \beta_\varepsilon(\tau) \leq 3/2. \tag{3.5}$$

Let now be M a constant such that, for $\varepsilon \in (0, \tilde{\varepsilon}]$,

$$|\beta_\varepsilon(\tau) - 1| \leq M\varepsilon, \quad |\beta'_\varepsilon(\tau)| \leq M\varepsilon. \tag{3.6}$$

Finally we define:

$$\tilde{w}_\varepsilon(\tau) = \cos\tau \exp\Big(-\varepsilon\tau + 2\varepsilon\int_0^\tau \varphi(s)\cos^2 s\, ds\Big), \qquad w_\varepsilon(\tau) = \tilde{w}_\varepsilon(\tau)e^{\varepsilon\tau}.$$

So, $\beta_\varepsilon(\tau)$ and $\tilde{w}_\varepsilon(\tau)$ are 2π-periodic C^∞ functions. Furthermore it is easy to see that w_ε is the solution of the Cauchy problem

$$w_\varepsilon''(\tau) + \beta_\varepsilon(\tau) w_\varepsilon(\tau) = 0, \quad w_\varepsilon(0) = 1, \ w_\varepsilon'(0) = 0. \tag{3.7}$$

ii) We are now ready to construct the coefficient $a(t)$ for $t \in [0, +\infty)$ as follows:

$$a(t) = \beta_{\tilde{\varepsilon}}(4\pi\nu_k t) \qquad \text{for } t \in I_k\,, \qquad k = 0, 1, \dots \tag{3.8}$$

where β_ε is given by (3.4) and the increasing sequence of integers ν_k will be chosen later. Clearly we obtain a function $a \in C^\infty([0, +\infty))$, periodic in each interval I_k; so (2.7) is satisfied with $P_k = 1/2\nu_k$, $l_1 = \min \beta_{\tilde{\varepsilon}}(\tau)$, $l_2 = \max \beta_{\tilde{\varepsilon}}(\tau)$.

Now we define a solution $u \in C^\infty([0, +\infty); \gamma^\sigma(\mathbf{T}))$ for any $\sigma > 1$ of $Lu = 0$ and take $u_0(x) = u(0, x)$, $u_1(x) = \partial_t u(0, x)$ as Cauchy data in (1.3). Let us set

$$u(t, x) = \sum_{k=1}^{\infty} v_k(t) e^{i\nu_k x}. \tag{3.9}$$

In order to have $Lu = 0$, we impose

$$v_k''(t) + \nu_k^2 a(t) v_k(t) = 0\,; \tag{3.10}$$

hence, if we impose

$$v_k(t_k) = 1, \ v_k'(t_k) = 0, \quad \text{for } t_k = k + 1/2\,, \tag{3.11}$$

we have, thanks to equation (3.7),

$$v_k(t) = w_{\tilde{\varepsilon}}(4\pi\nu_k(t - t_k)), \quad t \in I_k. \tag{3.12}$$

In particular

$$v_k(k) = e^{-2\pi\tilde{\varepsilon}\nu_k}, \quad v_k'(k) = 0, \tag{3.13}$$

$$v_k(k+1) = e^{2\pi\tilde{\varepsilon}\nu_k}, \quad v_k'(k+1) = 0. \tag{3.14}$$

Now, in order to estimate u for $t \le k$, we define the energy "of order k":

$$E_k(t) = |v_k'(t)|^2 + \nu_k^2 a(t) |v_k(t)|^2\,. \tag{3.15}$$

Differentiating (3.15) and using Gronwall inequality, from (3.13) and (3.10) we obtain, for $t \le k$:

$$\begin{aligned} E_k(t) &\le E_k(k) \exp\Big[\int_0^k |a'(t)|/a(t)dt\Big] \\ &= \nu_k^2 \exp\Big[-4\pi\tilde{\varepsilon}\nu_k + \sum_{j=1}^{k-1} \int_{I_j} |a'(t)|/a(t)dt\Big]. \end{aligned} \tag{3.16}$$

But, thanks to (2.3), (3.5), (3.6) and (3.8),

$$\int_{I_j} |a'(t)|/a(t)dt \le 8\pi M \nu_j \tilde{\varepsilon}, \tag{3.17}$$

so, finally, for $t \le k$, we obtain

$$E_k(t) \le \exp\Big[-4\pi\tilde{\varepsilon}\nu_k + 8\pi M\tilde{\varepsilon}\sum_{j=1}^{k-1}\nu_j + 2\log(\nu_k)\Big]. \tag{3.18}$$

Now we choose

$$\nu_k = \mu^k,$$

with μ an integer so large that, for $k \ge 2$, one has:

$$2\pi\tilde{\varepsilon}\nu_k \ge 8\pi M\tilde{\varepsilon}\sum_{j=1}^{k-1}\nu_j + 2\log(\nu_k). \tag{3.19}$$

From (3.18) and (3.19), we obtain, for $t \le k$:

$$E_k(t)\exp\big(\nu_k^{1/\sigma}\big) \le \exp\Big[-2\pi\tilde{\varepsilon}\nu_k + \nu_k^{1/\sigma}\Big], \tag{3.20}$$

and this expression goes to 0 for $k \to \infty$, for any $\sigma > 1$.

So, for u defined by (3.9), for any $\sigma > 1$ and for any $T_0 > 0$ we have $u \in C^\infty([0,T_0],\gamma^\sigma(\mathbf{T}))$, that is $u \in C^\infty([0,+\infty),\gamma^\sigma(\mathbf{T}))$. In particular $u(0,x)$ and $\partial_t u(0,x)$ are in $\gamma^\sigma(\mathbf{T})$ for any $\sigma > 1$.

On the other hand, from (3.14) immediately follows that

$$E_k(k+1)\exp\big(-\nu_k^{1/\sigma}\big) = \exp\Big[4\pi\tilde{\varepsilon}\nu_k - \nu_k^{1/\sigma}\Big], \tag{3.21}$$

and so, for any $\sigma > 1$ and any $T_1 > 0$, $u(t,\cdot)$ and $\partial_t u(t,\cdot)$ are not bounded in $(\mathcal{D}^\sigma(\mathbf{T}))'$ for $t \in [T_1,+\infty)$.

iii) We define now the new coefficient $a(t)$ for $t \in [0,+\infty)$ as follows:

$$a(t) = \beta_{\varepsilon_k}(4\pi\nu_k t) \qquad \text{for } t \in I_k\,, \qquad k = 0,1,\dots \tag{3.22}$$

where β_ε is given by (3.4), ε_k is a sequence decreasing to 0 and ν_k an increasing sequence of integers; these sequences will be chosen later, in relation to *modulus function* Ω verifying Definition 2.1; clearly we obtain a function $a \in C^\infty([0,+\infty))$, periodic in each interval I_k and going to 1 when $t \to +\infty$; so (2.9) is satisfied and again with $P_k = 1/2\nu_k$.

Let Ω be given as in (2.2); taking (3.22) into account, in order to have $a \in \Omega LL([0,+\infty))$ it will be sufficient to impose:

$$\varepsilon_k\,\nu_k = \log(\nu_k)\,\Omega(1/\nu_k)\,. \tag{3.23}$$

We define now a solution u of (1.3) having the form given by (3.9) with v_k verifying (3.10) and (3.11). Then (3.12), (3.13), (3.14) and (3.16) are satisfied with $\tilde{\varepsilon}$ substituted by ε_k, while in (3.17) $\tilde{\varepsilon}$ is substituted by ε_j. So estimate (3.18) becomes, for $t \le k$:

$$E_k(t) \le \exp\Big[-4\pi\varepsilon_k\nu_k + 8\pi M\sum_{j=1}^{k-1}\varepsilon_j\nu_j + 2\log(\nu_k)\Big]. \tag{3.24}$$

Now we choose:

$$\nu_k = 2^{B^k} \tag{3.25}$$

with B an integer sufficiently large to be chosen.

From (3.23) and (3.25) we have:

$$\varepsilon_k \, \nu_k = B^k \, \Omega\big(2^{-B^k}\big) \log 2 \,, \tag{3.26}$$

and so:

$$\frac{\varepsilon_{k+1} \, \nu_{k+1}}{\varepsilon_k \, \nu_k} = B \, \frac{\Omega\big(2^{-B^{(k+1)}}\big)}{\Omega\big(2^{-B^k}\big)}. \tag{3.27}$$

From (3.27) and (2.2), by choosing B sufficiently large, we obtain:

$$\pi \varepsilon_k \nu_k \geq 8\pi M \sum_{j=1}^{k-1} \varepsilon_j \nu_j. \tag{3.28}$$

Moreover we remark that, thanks to (2.2) and (3.23), surely we have, for any $k \geq 1$:

$$\pi \varepsilon_k \nu_k \geq 2 \log(\nu_k). \tag{3.29}$$

From (3.23), (3.24), (3.28) and (3.29), we obtain:

$$E_k(t) \, (\nu_k^s) \leq \exp\Big[\big(-2\pi \, \Omega(1/\nu_k) + s\big) \log(\nu_k)\Big]$$

and this expression, thanks again to (2.2), goes to 0 for $k \to +\infty$, for any $s \in \mathbf{R}$.

So, for u defined by (3.9), for any $s \in \mathbf{R}$ and for any $T_0 > 0$ we have $u \in C^\infty([0, T_0], H^s(\mathbf{T}))$, that is $u \in C^\infty([0, +\infty), H^s(\mathbf{T}))$. In particular $u(0,x)$ and $\partial_t u(0,x)$ are in $H^s(\mathbf{T})$ for any $s \in \mathbf{R}$.

On the other hand, in order to prove (2.5), we will use, instead of (3.15), the following energy (see [5] and [6]):

$$\tilde{E}_k(t) = |v_k'(t)|^2 + \nu_k^2 |v_k(t)|^2 \,. \tag{3.30}$$

From (3.14) with $\tilde{\varepsilon}$ substituted by ε_k, it immediately follows that:

$$\tilde{E}_k(k+1) = |v_k'(k+1)|^2 + \nu_k^2 |v_k(k+1)|^2 = \nu_k^2 e^{4\pi \varepsilon_k \nu_k} \,. \tag{3.31}$$

Differentiating (3.30) and using again the Gronwall inequality, from (3.31) and (3.10) we obtain, for $k+1 \leq t < +\infty$:

$$\begin{aligned} \tilde{E}_k(t) &\geq \tilde{E}_k(k+1) \exp\Big[-\nu_k \int_{k+1}^{+\infty} |1 - a(t)| \, dt\Big] \\ &= \nu_k^2 \exp\Big[4\pi \varepsilon_k \nu_k - \nu_k \sum_{j=k+1}^{+\infty} \int_{I_j} |1 - a(t)| \, dt\Big]. \end{aligned} \tag{3.32}$$

But, thanks to (2.3), (3.6) and (3.22),

$$\int_{I_j} |1 - a(t)| \, dt \leq M \varepsilon_j, \tag{3.33}$$

so, finally, for $t \geq k+1$, we obtain:

$$\tilde{E}_k(t)(\nu_k^{-s}) \geq \exp\Big[\nu_k\big(4\pi\varepsilon_k - M\sum_{j=k+1}^{+\infty}\varepsilon_j\big) + (2-s)\log(\nu_k)\Big]. \tag{3.34}$$

In order to evaluate from below the right-hand term in (3.34), we consider the ratio

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} = B\,\frac{\tau'\,\Omega(\tau')}{\tau''\,\Omega(\tau'')}, \tag{3.35}$$

where, for sake of simplicity, we have posed:

$$\tau' := \nu_{k+1}^{-1} = 2^{-B^{k+1}} \quad \text{and} \quad \tau'' := \nu_k^{-1} = 2^{-B^k}.$$

But, by the mean value theorem, we get for some $\tau \in [\tau', \tau'']$:

$$\frac{\tau'\,\Omega(\tau')}{\tau''\,\Omega(\tau'')} = \frac{\tau'}{\tau''}\Big[\frac{\Omega(\tau')-\Omega(\tau'')}{\Omega(\tau'')} + 1\Big] = \frac{\tau'}{\tau''}\Big[\frac{|\Omega'(\tau)|(\tau''-\tau')}{\Omega(\tau'')} + 1\Big]. \tag{3.36}$$

Remembering that Ω is a *modulus function* (see Definition 2.1. and, in particular, (2.2)), from (3.35) and (3.36) we obtain:

$$\frac{\varepsilon_{k+1}}{\varepsilon_k} \leq 2B\,\frac{\tau'\,|\Omega'(\tau)|\,\tau''}{\tau''\,\Omega(\tau'')} \leq 2B\,\frac{\tau'\,|\Omega'(\tau')|\,\tau''}{\tau''\,\Omega(\tau'')} \leq \frac{2B}{\Omega(\nu_k^{-1})}\,. \tag{3.37}$$

From (3.37) it follows immediately that, for $k \geq \overline{k}$,

$$2\pi\varepsilon_k \geq M\sum_{j=k+1}^{+\infty}\varepsilon_j$$

and so, from (3.34) and (3.23), we get, for $k+1 \leq t < +\infty$:

$$\begin{aligned}\tilde{E}_k(t)(\nu_k^{-s}) &\geq \exp[2\pi\varepsilon_k\nu_k + (2-s)\log(\nu_k)] \\ &= \exp\big[\log(\nu_k)\big(2\pi\Omega(1/\nu_k) + 2 - s\big)\big].\end{aligned} \tag{3.38}$$

From (3.38), (2.5) immediately follows. □

Remark 3.1. Let $\mathcal{H}(\mathbf{T}_\rho)$, $\rho > 0$, be the topological vector space of all functions $f(z)$, $z = x + iy$, analytic in the strip $\mathbf{R}_x \times \{|y| < \rho\}$ and 2π-periodic in the x-variable; moreover let $\mathcal{H}(\mathbf{T}_\rho)'$ be its dual space.

We remark that we have proved something more that the statement of Theorem 2.3. ii). In fact (3.20), with $\nu_k^{1/\sigma}$ substituted by $\rho\,\nu_k$, shows that the solution u belongs to $C^\infty([0,+\infty), \mathcal{H}(\mathbf{T}_\rho))$ for $\rho < 2\pi\tilde{\varepsilon}$. In particular $u(0,x)$ and $\partial_t u(0,x)$ are in $\mathcal{H}(\mathbf{T}_\rho)$. On the other hand, (3.21) shows that for any $\overline{t}$, $u(t,\cdot)$ is unbounded in $(\mathcal{H}(\mathbf{T}_\rho))'$ for t $\in (\overline{t}, +\infty)$ and again for $\rho < 2\pi\tilde{\varepsilon}$.

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Hyperbolic Problems and Regularity Questions
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On the Operator Splitting Method: Nonlinear Balance Laws and a Generalization of Trotter-Kato Formulas

Rinaldo M. Colombo and Andrea Corli

Abstract. Two different applications of the operator splitting method are presented here. The first one concerns hyperbolic systems of balance laws in one space dimension: we state the existence and the stability of solutions for initial data with bounded variation. As an example a case of vehicular traffic flow is then considered. The second application concerns abstract nonlinear semigroups in a metric space: we show how a composition of semigroups can be defined, thus generalizing Trotter-Kato product formulas to nonlinear semigroups.

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1. Introduction

The operator splitting method is well known to be a powerful technique whose many applications range from abstract semigroup theory, [15], to partial differential equations, [14], and to numerical schemes, [22]. Some recent results exploiting this method in two different settings are given here.

This method allows to *"sum"* two known operators, say S^1 and S^2, yielding a third one, Σ. Essentially, $\Sigma_t u$ is computed applying alternatively S^1 and S^2 for a time t/n and then letting $n \to +\infty$.

We first consider the solution operator Σ generated by a hyperbolic system of balance laws in one space dimension,

$$\partial_t u + \partial_x f(u) = g(t, x, u) \tag{1.1}$$

and obtain it as the sum of the semigroup S^1 generated by the homogeneous (or *convective*) equation

$$\partial_t u + \partial_x f(u) = 0 \tag{1.2}$$

with the flow S^2 generated by the differential system

$$\partial_t u = g\,(t, x, u)\ . \tag{1.3}$$

More precisely, we require below that (1.2) is of Temple type, [23], while very mild assumptions are made on the source term g. If there exists a domain which is invariant both for the homogeneous equation (1.2) and for the differential equation (1.3), then the initial-value problem for the full system (1.1) is well posed in the class of functions with bounded variation. Systems of balance laws satisfying the assumptions above arise in many mathematical models, [9]; we show here an application to vehicular traffic flow.

In the framework of differential equations, the operator splitting method is known also as *fractional step* method. In the proof of the well-posedness of (1.1) it works as follows. Fix a parameter $\varepsilon > 0$ and approximate to an order ε the initial data u_o with a piecewise constant function u_o^ε; an analogous approximation is done for the function $x \to g(t, \cdot, u)$, giving g^ε. Then (first half-step), at any jump point of u_o^ε solve the Riemann problem for the *homogeneous equation* (1.2) in an approximate way, that is, by letting the Rankine-Hugoniot conditions be satisfied to an order ε; call u^ε the approximate solution. If ε is sufficiently small then no wave interactions occur during a time interval $[0, \varepsilon]$, by finite propagation speed. Solve then (second half-step) the *ordinary differential equation* $\partial_t u = g^\varepsilon\,(t, x, u)$ with the trace $u^\varepsilon(\varepsilon, x)$ as initial data at time $t = 0$ and let $u^\varepsilon(\varepsilon, x)$ be the value of this solution at time ε. All that is the first loop of the algorithm, which is continued by solving again approximatively the equation (1.2) up to time 2ε with $u^\varepsilon(\varepsilon, x)$ as initial data at time $t = \varepsilon$ and so on. It is then proved that this algorithm can be extended to any positive time. Thanks to uniform estimates one can finally pass to the limit for $\varepsilon \to 0$ and prove the existence of a solution to (1.1). The well-posedness and a characterization of the solution, see [1, 9], can be proved as well. If g does not depend on t, then this limiting procedure yields a semigroup.

A natural question then arises, namely under which conditions two semigroups can be *combined* as above, resulting in a third, new, semigroup. In the linear case, this is a classical problem in the theory of (linear) semigroups, [15].

The second part of this note is devoted to related results in the *nonlinear* case, [10]. The framework is provided by complete metric space, so that the whole construction is fully nonlinear. The *operator splitting* method considered here is defined in (4.1). Under suitable conditions, two semigroup S^1 and S^2 that approximately commute define a third semigroup $S^1 \oplus S^2$. By *"approximately commute"* we mean here that $d(S_t^1 S_t^2 u, S_t^2 S_t^1 u) = \mathcal{O}(t^2)$ for $t \to 0$, while the sufficient conditions on the semigroups are the local Lipschitz dependence on u and a sort of finite propagation speed, see Definition 4.1 below.

2. Hyperbolic balance laws

Let Ω be the closure of a non empty, open and connected subset of $\mathbb{R}^n$. We consider the system of conservation laws (1.2) for $t \geq 0$ and $x \in \mathbb{R}$. We assume that this system is of Temple type, i.e.,

(T) the function $f\colon \Omega \mapsto \mathbb{R}^n$ is smooth; the Jacobian matrix $Df(u)$ admits n real distinct eigenvalues; shock and rarefaction curves coincide; there is a set of Riemann coordinates.

The reader is referred to [6, 13] for the basic definitions on hyperbolic conservation laws. In particular the hypothesis above on the eigenvalues of Df is often referred to as the *strict hyperbolicity* condition. The assumption on the shock-rarefaction curves is satisfied, for instance, if each the characteristic field either has straight lines as integral curves or it is linearly degenerate. Systems satisfying this condition were introduced in [23]. At last, the local existence of a set of Riemann coordinates is equivalent to Frobenius involutive condition. In this setting, general results on the well-posedness of the system (1.2) were proved in [4, 5].

On the source term g in (1.1) we assume

(S) $g\colon\ [0,+\infty[\, \times \mathbb{R} \times \Omega \mapsto \mathbb{R}^n$ is measurable in (t,x,u) and smooth in u; there exists a positive finite measure μ on $\mathbb{R}$ such that for a.e. $x_1, x_2 \in \mathbb{R}$ with $x_1 \leq x_2$

$$\left| g(t, x_1^-, u) - g(t, x_2^+, u) \right| \leq \mu\left([x_1, x_2]\right)\ ; \tag{2.1}$$

there exist functions $A, B \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}\left([0,+\infty[\,;\mathbb{R}\right)$ and for every compact set $K \subseteq \Omega$ there exist $L_K \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}\left([0,+\infty[\,;\mathbb{R}\right)$ such that for a.e. $t \in [0,+\infty[$, a.e. $x \in \mathbb{R}$

$$|g(t,x,u_2) - g(t,x,u_1)| \leq L_K(t)\,|u_2 - u_1| \quad \text{for } u_1, u_2 \in K, \tag{2.2}$$

$$|g(t,x,u)| \leq A(t) + B(t)|u| \quad \text{for } u \in \Omega\,. \tag{2.3}$$

Next, our last assumption on (1.1) is a compatibility condition between the two problems (1.2) and (1.3). A set $\mathcal{U} \subseteq \Omega$ is positively invariant for (1.2) if for any initial data u_o with range in $\mathcal{U}$ the corresponding solution u takes values at all subsequent times in $\mathcal{U}$, as long as it exists. The definition of invariance of $\mathcal{U}$ with respect to the ordinary differential system (1.3) parameterized by $x \in \mathbb{R}$ is analogous. Invariant domains both for conservation laws and ordinary differential equations have been studied separately since a long time, see for instance [16, 20]. We require that:

(C) there exists a domain $\mathcal{U} \subseteq \Omega$ that is positively invariant both for the conservation law (1.2) and for the ordinary differential equation (1.3).

The space of functions defined on $\mathbb{R}$ with values in $\mathcal{U}$ and having bounded total variation is denoted by $\mathbf{BV}(\mathbb{R},\mathcal{U})$, while $\mathrm{TV}(u)$ denotes the total variation of a function $u \in \mathbf{BV}(\mathbb{R},\mathcal{U})$. Moreover we say that a function u belongs to the space $\mathbf{L}^{\mathbf{1}}_{*}(\mathbb{R},\mathcal{U})$ if $u \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbb{R},\mathcal{U})$ and there exist u_+, u_- in $\mathcal{U}$ such that $u - u_+$

(respectively $u - u_-$) is summable at $+\infty$ ($-\infty$). We define then

$$\mathbf{X}(\mathbb{R},\mathcal{U}) = \mathbf{L}^1_*(\mathbb{R},\mathcal{U}) \cap \mathbf{BV}(\mathbb{R},\mathcal{U}) . \tag{2.4}$$

Theorem 2.1. *Assume conditions* ($\mathbf{T}$), ($\mathbf{S}$) *and* ($\mathbf{C}$). *Then, for every initial data* $u_o \in \mathbf{X}(\mathbb{R},\mathcal{U})$ *the Cauchy problem*

$$\begin{cases} \partial_t u + \partial_x f(u) = g(t,x,u) \\ u(0,x) = u_o(x) \end{cases} \tag{2.5}$$

admits a solution u *with* $u(t) \in \mathbf{X}(\mathbb{R},\mathcal{U})$ *for a.e.* $t \in [0,+\infty[$. *Moreover, for all* $M,T > 0$ *there exists a positive constant* L *such that if* $u_o, u_o' \in \mathbf{X}(\mathbb{R},\mathcal{U})$ *and* $\mathrm{TV}(u_o), \mathrm{TV}(u_o') \leq M$, *then the corresponding solutions* u, u' *satisfy for all* $t \in [0,T]$

$$\|u(t) - u'(t)\|_{\mathbf{L}^1} \leq L \cdot \|u_o - u_o'\|_{\mathbf{L}^1} . \tag{2.6}$$

The proof of Theorem 2.1 as well as further estimates on the solutions, is given in [3]. Note that (2.6) is relevant only when u_o and u_o' are such that $u_o - u_o' \in \mathbf{L}^1(\mathbb{R},\mathbb{R})$. Otherwise, obviously, no Lipschitz regularity as (2.6) may hold. Moreover, the choice of the space $\mathbf{L}^1_*$ is motivated by applications to cases where the initial data cannot realistically be assumed in $\mathbf{L}^1(\mathbb{R},\mathcal{U})$. For instance, in the case of vehicular traffic flows, the *"limits"* of the traffic density at $\pm\infty$ represent the asymptotic inflow and outflow of the considered road, see the example below. We refer also to [9] for more detailed results under slightly stronger assumptions on the source term as well as for a proof of the uniqueness of solutions.

The main tool in the proof of the theorem above is the *operator splitting method*, which works as follows. We construct first for any $\varepsilon > 0$ an approximate semigroup $S^\varepsilon \colon [0,+\infty[\times \mathcal{D}^\varepsilon \to \mathcal{D}^\varepsilon$ to (1.2), for suitable domains $\mathcal{D}^\varepsilon$ of discretized data. We consider then an ε-approximation of equation (1.3) and prove the existence of a solution process $\Sigma^\varepsilon \colon [0,+\infty[\times \mathcal{D}^\varepsilon \to \mathcal{D}^\varepsilon$ generated by it. From these two operators we define, for all $k \in \mathbb{N}$

$$F^\varepsilon_{0,t} u = \begin{cases} S^\varepsilon_t u & \text{if } t \in [0,\varepsilon[\,, \\ \Sigma^\varepsilon_{0,\varepsilon}\left(S^\varepsilon_\varepsilon u\right) & \text{if } t = \varepsilon, \\ S^\varepsilon_{t-k\varepsilon}\left(\bigcirc_{i=1}^{k-1} F^\varepsilon_{i\varepsilon,(i+1)\varepsilon}\right) F^\varepsilon_{0,\varepsilon} u & \text{if } t \in [k\varepsilon,(k+1)\varepsilon[\ , \end{cases} \tag{2.7}$$

where we denote

$$\bigcirc_{i=1}^{k-1} F^\varepsilon_{i\varepsilon,(i+1)\varepsilon} = F^\varepsilon_{\varepsilon,2\varepsilon} \circ F^\varepsilon_{2\varepsilon,3\varepsilon} \circ \cdots \circ F^\varepsilon_{(k-1)\varepsilon,k\varepsilon} .$$

The operator F^ε is proved to be well defined for all $t \geq 0$ in the domain $\mathcal{D}^\varepsilon$. Its limit for $\varepsilon \to 0$ exists and gives the solution to (1.1).

3. An application to traffic flow

Some continuum conservative models for vehicular traffic flow, [2, 7] can be written in the form

$$\begin{cases} \partial_t \rho + \partial_x \left(\rho \cdot v(\rho,y)\right) = 0 \\ \partial_t y + \partial_x \left(y \cdot v(\rho,y)\right) = 0 \end{cases} \tag{3.1}$$

where ρ is the car density, v the speed and y a flow variable, usually originated by some analogy with the linear momentum typical of fluid dynamic. These systems are known under the name of Keyfitz-Kranzer systems, [17]. If v is a smooth function and $\rho\,\partial_\rho v + y\,\partial_y v \neq 0$ then they are in the Temple class.

As a simple example of source terms we consider the following model for entries/exits in a highway, where the convective part is given by [7]:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = a_{\mathrm{in}}(t,x)\left(1 - \dfrac{\rho}{R}\right) - a_{\mathrm{out}}(t,x)\dfrac{\rho}{R} \\ \partial_t q + \partial_x\left((q-q_*)v\right) = -\left(\dfrac{a_{\mathrm{in}}(t,x)}{R} + \dfrac{a_{\mathrm{out}}(t,x)}{R}\right)(q-q_*). \end{cases} \tag{3.2}$$

Here q is a (weighted) momentum, q_* an "equilibrium" momentum, assumed to be constant, R the maximal car density, $v = \left(1 - \frac{\rho}{R}\right)\cdot\frac{q}{\rho}$. For what concerns the source terms, $a_{\mathrm{out}}(t,x) = g_{\mathrm{out}}(t)\chi(x)$, for $g_{\mathrm{out}}(t)$ the fraction of the traffic density per unit time that exits along $[a,b]$ and χ the characteristic function of the interval $[a,b]$. The term $a_{\mathrm{in}}(t,x)$ is defined analogously. We refer to [11] for a strategy to determine the parameters q_* and R in a dynamical way.

The source term in the first equation in (3.2) is rather classical and similar, for instance, to the analogous term in [18]. On the contrary, the present choice of the right hand side of the second equation in (3.2) is less traditional. Indeed, its motivation is specific to the present model. The role of q_* is strictly related to *wide jams*, see [8]. These are persistent phenomena and consist in square waves with high traffic density travelling along the road. It has been observed that in correspondence of entries and exits drivers tend to stabilize the traffic flow, reducing the variation in speed among different vehicles. Correspondingly, the source terms in the second equation in (3.2) model this stabilization about the value q_* of the weighted flow.

Invariant sets for the conservative part in (3.2) are

$$\mathcal{U} = \left\{(\rho,q) \in [0,R]\times\mathbb{R} \colon v(\rho,q)\in[V_1,V_2],\ q_* + \frac{q-q_*}{\rho}R \in [Q_1,Q_2]\right\} \tag{3.3}$$

where $0 \le V_1 \le V_2 \le +\infty$ and $0 \le Q_1 \le Q_2 \le +\infty$, see Figure 1.

To check the invariance with respect to the ordinary differential equation (1.3) one can use the classical Nagumo condition, [20], which requires that the source term on the boundary of the domain points toward the domain. Then, for instance, if in (3.2) we have $a_{\mathrm{in}} = 0$ (no entry), then the choice $V_1 = 0$, $V_2 = +\infty$ makes the domain (3.3) invariant for (3.2), for any Q_1, Q_2. On the other hand, if $a_{\mathrm{out}} = 0$ (no exit) then the choice $V_1 = 0$, $Q_1 = 0$ gives an invariant domain if $V_2 \ge Cq_*/R$, for a suitable constant $C > 0$. For other cases we refer to [3].

We can assume that at a given time t_o the traffic is described by suitable functions $\rho_o(x)$, $q_o(x)$ in an interval containing $[a,b]$, and set equal to constants $\rho_{o,-\infty}$, $q_{o,-\infty}$, $(\rho_{o,+\infty}$, $q_{o,+\infty})$ on the left of a (resp. on the of right of b). Under mild assumptions on the functions g_{in} and g_{out} Theorem 2.1 applies: solutions exist

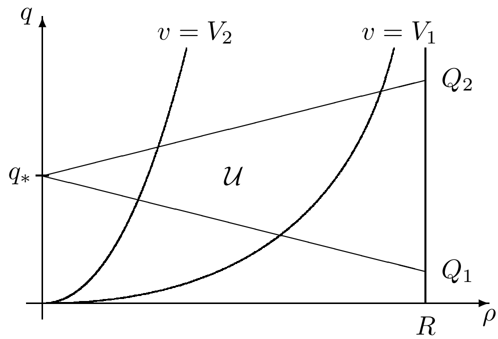

FIGURE 1. Invariant domain for the conservative part of (3.2).

and are valued in $\mathcal{U}$. For many more applications to traffic flows and numerical simulation we refer to [3].

Another interesting application of the operator splitting technique concerns multi-lane models for traffic flows, [12]. We point out that in this case the convective part is hyperbolic, albeit not strictly. However the particular structure of the model makes it possible to follow the same lines of the proof sketched above.

4. Operator splitting in an abstract framework

In this section, motivated by the methods of Section 2, we focus on the following general problem: *when and how, in a metric space, is it possible to combine two semigroups to obtain a third semigroup?* For brevity, motivated by the previous example, we refer to this combination of semigroups as to their *"sum"*. The choice of metric spaces as a framework for the present construction is motivated by the nonlinear nature of (1.1).

The literature dealing with definition and properties of sums of linear semigroups is very wide. Let us only mention [15] as a general reference.

Let us mention that a generalization of ordinary differential equations to metric spaces were introduced in [21] in the 80s to study discontinuous ordinary differential equations. In the case under consideration in the previous sections, one could wonder whether it is really necessary to use the operator splitting technique relying on *approximate* semigroups and whether it is possible to deduce some structural properties through the direct use of the *exact* semigroups.

We begin now the construction of the sum of two semigroups selecting a class of sufficiently regular semigroups.

Definition 4.1. Let $(\mathcal{U}, d)$ be a metric space. We denote by $\mathcal{S}(\mathcal{U})$ the set of all semigroups $S\colon [0, +\infty[\times \mathcal{U} \to \mathcal{U}$, i.e., $S_0 = \mathrm{Id}$, $S_s \circ S_t = S_{s+t}$, with the properties:

(S1) for every S and $T > 0$, $R > 0$, $u_o \in \mathcal{U}$, there exists $K = K(T, R, u_o)$ such that $d(S_t u, u) \leq Kt$ for $t \in [0, T]$ and $u \in \mathcal{U}$, $d(u, u_o) < R$;

(S2) for every S there exists a constant C such that $d(S_t u, S_t w) \leq e^{Ct} d(u,w)$ for $t > 0$ and $u, w \in \mathcal{U}$.

If $\mathcal{U}$ is a Banach space and S is linear, then **(S1)** is a local version of the uniform continuity of S while **(S2)** is the quasicontractivity, [15].

Many classes of ordinary differential equations have solution semigroups in the class $\mathcal{S}(\mathcal{U})$. Other examples are provided by scalar conservation laws (also in several space dimensions), where $\mathcal{U} = \mathbf{L}^\infty(\mathbb{R}^m) \cap \mathbf{L}^1(\mathbb{R}^m)$ with the $\mathbf{L}^1$ metric, see [13, Section 6.4]. In these cases **(S1)** is the finite propagation speed property, **(S2)** the Lipschitz continuous dependence from the initial data. For hyperbolic systems of conservations laws in one space dimension, the existence of a contractive semigroup is proved in suitable domains containing, at least, all functions with sufficiently small total variation, [4, 6]. In this case, contractivity is proved with respect to an *ad hoc* metric constructed on each specific conservation law, which is equivalent to the one induced by the $\mathbf{L}^1$ norm.

In the present setting, the operator splitting method takes the following form:

Definition 4.2. Let $S^1, S^2 \in \mathcal{S}(\mathcal{U})$, $\varepsilon > 0$. We define $\Sigma^{21,\varepsilon}\colon\ [0,+\infty[\,\times\,\mathcal{U} \mapsto \mathcal{U}$ by

$$\Sigma_t^{21,\varepsilon} u = \begin{cases} S_t^1 u & \text{for } t \in [0,\varepsilon[\\ S^1_{t-n\varepsilon} \bigcirc_{i=1}^n S^2_\varepsilon S^1_\varepsilon u & \text{for } t \in [n\varepsilon, (n+1)\varepsilon[\,,\ n \in \mathbb{N}\,. \end{cases} \tag{4.1}$$

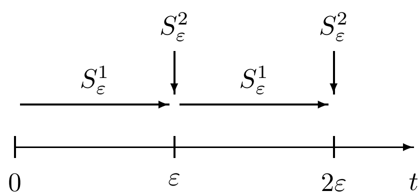

FIGURE 2. The first steps in the construction of $\Sigma_t^{21,\varepsilon}$.

Remark that if $S^2 = \mathrm{Id}$, i.e., $S_t^2 u = u$ for all t, u, then $\Sigma_t^{21,\varepsilon} = S_t^1$ for all ε and $t \geq 0$. Similarly, if $S^1 = \mathrm{Id}$ then $\Sigma_{n\varepsilon}^{21,\varepsilon} = S^2_{n\varepsilon}$ for any ε and n. Thus, the identity plays as the zero of the sum between semigroups. Moreover if $S^1 = S^2 = S$, then $\Sigma_t^{21,\varepsilon} = S_{2t}$ for every ε and n. Therefore, $S \oplus S = 2S$, meaning that $(2S)_t u = S_{2t} u$.

A schematic diagram of the splitting is given in Figure 2. We refer to [10] for different iterative constructions, for instance using the Strang splitting, [22]. The present construction yields the same result, so that the various properties hold also in those cases.

In the following for a semigroup $S^i \in \mathcal{S}(\mathcal{U})$, we denote by C_i and K_i the corresponding constants in **(S1)**–**(S2)**.

Theorem 4.3. *Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\varepsilon_n \to 0$ for $n \to \infty$. If $\Sigma_t^{21,\varepsilon_n} u$ converges pointwise to $\Sigma_t^{21} u$ as $n \to \infty$ for all $(t,u) \in [0,+\infty[\,\times\,\mathcal{U}$, then*

1. *the semigroup* Σ^{21} *satisfies* **(S1)** *and* **(S2)** *with* $C^{21} = C_1 + C_2$ *and* $K^{21} = K_1 + K_2$*;*
2. $\Sigma^{12,\varepsilon_n}$ *converges to the same limit* Σ^{21} *for all* $t \in [0, +\infty[$ *and* $u \in \mathcal{U}$*.*

The above result reflects the situation of conservation laws, where the existence of the limit $\lim_{\varepsilon\to 0} \Sigma^{21,\varepsilon}$ often follows by compactness arguments. The uniqueness of the limit and its properties then follow exploiting the specific structure of conservation laws.

Theorem 4.3 ensures some properties of Σ^{21}, provided the limit exists. To prove that the limit does exists a commutation condition is required. Until now no such condition on the semigroups S^1 and S^2 was assumed. To define the sum $S^1 \oplus S^2$ of two semigroups S^1 and S^2 we require that their commutator $d\left(S^1_t S^2_t u, S^2_t S^1_t u\right)$ vanishes as $t \to 0$ faster than what the Lipschitz dependence on t and u would ensure. So we give the following condition:

(C) there exists a function $\omega\colon [0,1] \mapsto [0, +\infty[$, with $\lim_{t\to 0+} \omega(t) = 0$, such that for any $T \in [0,1]$, $R > 0$, $u_o \in \mathcal{U}$ there exists $H = H(T, R, u_o)$ such that for every $t \in [0, T]$ and $u \in \mathcal{U}$, $d(u, u_o) < R$, it holds

$$d\left(S^1_t S^2_t u, S^2_t S^1_t u\right) \leq H \cdot t \cdot \omega(t)\,.$$

The conditions **(S1)** and **(S2)** imply only the first order bound $d\left(S^1_t S^2_t u, S^2_t S^1_t u\right) \leq Ht$ on the commutator. Condition **(C)** holds for instance if $\mathcal{U}$ is a Banach space and S^1, S^2 are quasicontractive uniformly continuous linear semigroups. We refer to [10] for examples where **(C)** fails.

Condition **(C)** is sufficient to ensure the the existence of the limit $\lim_{\varepsilon\to 0} \Sigma^{21,\varepsilon}$, as is stated in the next proposition.

Proposition 4.4. *Let* S^1, S^2 *in* $\mathcal{S}(\mathcal{U})$ *satisfy* (**C**)*. Fix* $\varepsilon > 0$*, define* $\varepsilon_n = \varepsilon 2^{-n}$ *and assume*

$$\sum_n \omega(\varepsilon_n) < \infty\,. \tag{4.2}$$

Let $\Sigma^{21,\varepsilon}_t$ *be as in* (4.1)*. Then, for every* $(t, u) \in [0, +\infty[\, \times\, \mathcal{U}$*, the sequence* $\Sigma^{\varepsilon_n}_t u$ *has a limit* $\Sigma_t u$ *in* $\mathcal{U}$*.*

We point out that condition (4.2) holds for any $\omega(t) \leq t^\alpha$, with $\alpha > 0$. Hence, the bound on the commutator required by Proposition 4.4 is weaker than the one required in [19] in the *linear* case.

In the previous statements the limit Σ^{21} may in general depend on the sequence ε_n used to pass to the limit. Therefore, to define the sum $S^1 \oplus S^2$ the existence of the limit $\lim_{\varepsilon\to 0} \Sigma^{21,\varepsilon}$ is not sufficient. The sum needs to be intrinsic and, in particular, independent from the choice of ε_n. To this aim, we need a condition stronger than **(C)**:

(C*) for any $T \in [0,1]$, $R > 0$, $u_o \in \mathcal{U}$ there exists $H = H(T, R, u_o)$ such that for every $t \in [0, T]$ and $u \in B(u_o, R)$

$$d\left(S^1_t S^2_t u, S^2_t S^1_t u\right) \leq H \cdot t^2\,.$$

The main theorem of this section now follows.

Theorem 4.5. *Let $\mathcal{U}$ be a complete metric space. Let S^1, S^2 be two semigroups in $\mathcal{S}(\mathcal{U})$ satisfying* **(C*)** *and $\Sigma_t u$ be as in Proposition* 4.4*. Then, Σ does not depend on ε and for all $u \in \mathcal{U}$,*

$$\lim_{t \to 0} \frac{1}{t} d\left(S_t^2 S_t^1 u, \Sigma_t u\right) = 0 \,. \tag{4.3}$$

We can now finally define the sum of two semigroups.

Definition 4.6. Let $\mathcal{U}$ be a complete metric space. Let S^1, S^2 be two semigroups in $\mathcal{S}(\mathcal{U})$ satisfying **(C*)** and fix $\varepsilon > 0$. Then we define the map $S^1 \oplus S^2 \in \mathcal{S}(\mathcal{U})$ by

$$(S^1 \oplus S^2)_t u := \lim_{n \to \infty} \Sigma_t^{\varepsilon_n} u \,.$$

It is now simple to check that the usual properties of the sum are verified. For example, commutativity and associativity are immediate. Moreover, denote by $\mathcal{R}(\mathcal{U})$ the subset of $\mathcal{S}(\mathcal{U})$ consisting of the reversible semigroups: $S \in \mathcal{R}(\mathcal{U})$ if there exists a semigroup $\bar{S} \in \mathcal{S}(\mathcal{U})$ satisfying $\bar{S}_t \circ S_t = \mathrm{Id}$ for all t. If $S \in \mathcal{R}(\mathcal{U})$, it is natural to denote, with a slight abuse of notation, $\bar{S}_t = S_{-t}$. Then, as a consequence of the definition above, $S \oplus \bar{S} = 0$ and $\bar{S} \oplus S = 0$ for any $S \in \mathcal{R}(\mathcal{U})$.

A natural definition of multiplication of semigroups by nonnegative scalars is as follows: $(\lambda S)_t = S_{\lambda t}$. It is then immediate to verify that $nS = \bigoplus_{i=1}^n S$.

We refer to [10] for applications of this semigroup sum to various partial differential equations. Here, we note that condition **(C*)** is too strong to allow the proof of Theorem 2.1 by means of Theorem 4.5.

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Subelliptic Estimates for some Systems of Complex Vector Fields

Makhlouf Derridj

1. Introduction

Our aim is to give some results on subellipticity for some systems of complex vector fields defined on an open set in $\mathbb{R}^n$. When one has a system of smooth real vector fields $(X_1, \ldots, X_r)$, the famous result of Hörmander [2] (with a precise coefficient of subellipticity by Rothschild and Stein [8]) gave a sufficient condition in terms of the Lie algebra generated by the vector fields X_j.

The situation is different in the case of a system of complex vector fields $(L_1, ., L_r)$. If one writes: $L_j = X_j + iY_j$, with X_j, Y_j, real, one may have the following inequality, called *maximal*, in L^2-norms:

$$\sum_{j=1}^{r} \Big(||X_j u|| + ||Y_j u|| \Big) \leq C \Big(\sum_{j=1}^{r} ||L_j u|| + ||u|| \Big) \quad \forall u \in \mathcal{D}(\Omega). \tag{1.1}$$

If the inequality (1.1) holds and each L_j is smooth, one can use Hörmander's condition on the real system $(X_1, ., X_r, Y_1, ., Y_r)$, to obtain a subelliptic estimate:

$$||u||_\epsilon \leq C \Big(\sum_{j=1}^{r} ||L_j u|| + ||u|| \Big) \quad \forall u \in \mathcal{D}(\Omega) \tag{1.2}$$

where Ω is the open set under consideration. But the inequality (1.2) may be true and (1.1) not. Results on (1.1) may be seen in the bibliography of [1]. When $r = 1$, results on (1.2) are obtained in [3, 9].

In the case $r > 1$, we consider in a neighborhood of 0 in $\mathbb{R}^{n+1}$, the following vector fields

$$\begin{cases} L_j = \dfrac{\partial}{\partial t_j} + i \dfrac{\partial \varphi}{\partial t_j} \dfrac{\partial}{\partial x}, \ j = 1, \ldots, n, \ , t \in \mathbb{R}^n, \ x \in \mathbb{R} \\ \varphi \in C^1(\Omega, \mathbb{R}), \ 0 \in \Omega \subset \mathbb{R}^n, \varphi = \varphi(t), \varphi(0) = 0 \end{cases} \tag{1.3}$$

We search for sufficient conditions on φ, for the existence of $\epsilon > 0$, $0 \in \omega \subset \Omega$, $0 \in I \subset \mathbb{R}$ such that

$$||u||_\epsilon \leq C\left(\sum_{j=1}^{n} ||L_j u|| + ||u||\right) \quad \forall u \in \mathcal{D}(\omega \times I) \tag{1.4}$$

In fact, we make a finer study, obtaining microlocal subellipticity : if (η, ξ) denote the dual variables of (t, x), we see from (1.3) that the system (L_j) is elliptic in the directions (η, ξ) with $\eta \neq 0$. So one has to study subellipticity in conic neighborhoods of $(\xi > 0, \eta = 0)$ and $(\xi < 0, \eta = 0)$.

We mention that H. Maire studied these problems in [5, 6] particularly in norms uniform in the t-variables. Let us mention also that J. Nourrigat did a lot of work in this subject, as for example in [7].

Finally we want to thank B. Helffer and H. Maire for numerous discussions on this subject.

2. Expression of u in terms of $(L_1 u, \ldots, L_n u)$

To simplify, we write:

$$L_j u = f_j, \;\; j = 1, \ldots, n \;\text{ or }\; Lu = f\,, u \in \mathcal{D}(\Omega \times \mathbb{R}) \tag{2.1}$$

Then using partial Fourier transform in x, one has:

$$\frac{\partial \widehat{u}}{\partial t_j}\,(t, \xi) - \xi \frac{\partial \varphi}{\partial t_j}(t)\; \widehat{u}\;(t, \xi) = \widehat{f_j}(t, \xi). \tag{2.2}$$

Now, consider a neighborhood of 0 denoted by ω with $\bar{\omega} \subset \Omega$ and let γ_t be a piecewise smooth curve such that

$$\gamma_t(0) = t \in \omega, \;\; \gamma_t(1) \notin \omega, \;\; \gamma_t : [0, 1] \to \Omega\,. \tag{2.3}$$

Then we can integrate the system (2.2) along the curve γ_t and, denoting $\gamma_t : \tau \in [0, 1] \to \Omega$

$$\widehat{u}(t, \xi) = -\int_0^1 \exp\;\left[\xi \cdot \Big(\varphi(t) - \varphi\Big(\gamma(\tau)\Big)\Big)\right] \widehat{f}\Big(\gamma(\tau), \xi\Big)\;\cdot \gamma'(\tau) d\tau \tag{2.4}$$

This will be the basic formula to study microlocal subellipticity in $(\xi > 0, \eta = 0)$ or $(\xi < 0, \eta = 0)$ directions.

3. Microlocal subellipticity in a conic neighborhood of $(\xi > 0, \eta = 0)$

We try now to give a sufficient condition for microlocal subellipticity in the positive direction $\xi > 0$. The condition (H_1) we state looks somehow abstract. We shall give in Sections 5, 6, 7 various types of examples.

The hypothesis (H_1)

1) There exist a neighborhood ω of 0 with $\bar{\omega} \subset \Omega$ and a finite number of subsets of ω denoted by $\omega_1, \ldots, \omega_k$ such that $\omega \setminus \cup_{j=1}^{k} \omega_j$ has measure 0;
2) $\forall j \in \{1, \ldots, k\}$, there exists $\gamma_j : \omega_j \times [0,1] \to \Omega$ with the following properties: $\forall\, t \in \omega_j$, the curve $\gamma_j(t, .)$ has finite C^1- pieces and:
   i) $\gamma_j(t, 0) = t,\ \gamma_j(t, 1) \notin \omega,\ \forall\, t \in \omega_j$;
   ii) γ_j is C^1, outside a negligible set E and satisfies:

$$\begin{cases} |\gamma_j'| = |\dfrac{\partial \gamma_j}{\partial \tau}| \leq c_2 \ ;\ 0 < c_1 \leq |\det(D_t \gamma_j)| \leq c_2 \ \text{ on } \complement E \\ \varphi\Big(\gamma_j(t,\tau)\Big) - \varphi(t) \geq c_1 \tau^{\alpha} \ \ (t,\tau) \in \omega_j \times [0,1] \\ \text{where } c_1, c_2 \text{ and } \tau \text{ are positive constants} \end{cases} \tag{3.1}$$

The second inequality in (3.1), will give the gain of subellipticity equal to $\dfrac{1}{\alpha}$. In the classes of examples given after Section 5, the constant α, will be an integer simply related to the function φ.

Theorem 3.1. *Assume the hypothesis* (H_1) *be satisfied. Then there exists* $C > 0$ *such that*

$$\int_{\omega \times \mathbb{R}^+} \xi^{2/\alpha} |\widehat{u}(t,\xi)|^2 dt d\xi \leq C \int_{\omega \times \mathbb{R}^+} |\widehat{f}(t,\xi)|^2 dt d\xi, \qquad \forall\, u \in \mathcal{D}(\omega \times I).$$

Proof. Fix $t \in \omega_j$. From (H_1), and using Cauchy-Schwarz inequality in (2.4), we obtain:

$$\begin{aligned} |\widehat{u}(t,\xi)|^2 \leq & \int_0^1 e^{\xi \cdot \left[\varphi(t) - \varphi\left(\gamma_j(t,\tau)\right)\right]} d\tau \\ & \times \int_0^1 e^{\xi \cdot \left[\varphi(t) - \varphi\left(\gamma_j(t,\tau)\right)\right]} \left| \widehat{f}\Big(\gamma_j(t,\tau), \xi\Big) \cdot \gamma_j'(t,\tau) \right|^2 d\tau \\ |\widehat{u}(t,\xi)|^2 \leq & \ c_2^2 \int_0^1 e^{-c_1 \tau^{\alpha} \xi} d\tau \int_0^1 e^{-c_1 \tau^{\alpha} \xi} \left| \widehat{f}\Big(\gamma_j(t,\tau), \xi\Big) \right|^2 d\tau \end{aligned} \tag{3.2}$$

So, integrating in t, one has:

$$\int_{\omega_j} |\widehat{u}(t,\xi)|^2 dt \leq c_2^2 \int_0^1 e^{-c_1 \tau^{\alpha} \xi} d\tau \int_0^1 \int_{t \in \omega_j} e^{-c_1 \tau^{\alpha} \xi} |\widehat{f}\big(\gamma_j(t,\tau), \xi\big)|^2 dt d\tau \tag{3.3}$$

Consider now the integral in the right and put $v = \gamma_j(t,\tau), \tau = \tau$.

Then from (3.1) one has:

$$\begin{aligned} \int_0^1 \int_{\omega_j} e^{-c_1 \tau^{\alpha} \xi} |\widehat{f}\big(\gamma_j(t,\tau), \xi\big)|^2 dt d\tau & \leq c_1^{-1} \int_0^1 \int_{v \in \Omega} e^{-c_1 \tau^{\alpha} \xi} |\widehat{f}(v,\xi)|^2 dv d\tau \\ & \leq c_1^{-1} \int_0^1 e^{-c_1 \tau^{\alpha} \xi} d\tau \int_{\omega} |\widehat{f}(v,\xi)|^2 dv \end{aligned} \tag{3.4}$$

So, finally, one has

$$\int_{\omega_j} |\widehat{u}(t,\xi)|^2 dt \le C_1^{-1} c_2^2 \left(\int_0^1 e^{-c_1 \tau^\alpha \xi} d\tau \right)^2 \int |\widehat{f}(v,\xi)|^2 dv \tag{3.5}$$

Now, we have the inequality:

$$\int_0^1 e^{-c_1 \tau^\alpha \xi} d\tau = \int_0^{\xi^{1/\alpha}} e^{-c_1 s^\alpha} \xi^{-1/\alpha} ds \le c_3\, \xi^{-1/\alpha} \ , \ c_3 > 0 \tag{3.6}$$

From (3.5) and (3.6) we deduce:

$$\int_{\omega_j} |\widehat{u}(t,\xi)|^2 dt \le c_1^{-1} c_2^2 c_3\, \xi^{-2/\alpha} \int_\omega |\widehat{f}(v,\xi)|^2 dv, \quad \forall u \in \mathcal{D}(\omega \times I) \tag{3.7}$$

□

4. Microlocal subellipticity in a conic neighborhood of $(\xi < 0, \eta = 0)$

Now we state the hypothesis (H_2), which is the analogue of (H_1), in the case of the negative cone.

Hypothesis (H_2)
The only thing different from (H_1) is the second inequality of (3.1). So (3.1) is replaced by:

$$\begin{cases} |\gamma_j'| = |\dfrac{\partial \gamma_j}{\partial \tau}| \le c_2\,;\ 0 < c_1 \le |\det(D_t \gamma_j)| \le c_2 \quad \text{on} \quad \complement E \\ \varphi\big(\gamma_j(t,\tau)\big) - \varphi(t) \le -c_1 \tau^\alpha \in \omega_j \times [0,1] \end{cases} \tag{4.1}$$

Theorem 4.1. *Assume that hypothesis (H_2) satisfied. Then there exists a constant $C > 0$ such that:*

$$\int_{\omega \times \mathbb{R}^-} |\xi|^{2/\alpha} |\widehat{u}(t,\xi)|^2 dt d\xi \le C \int_{\omega \times \mathbb{R}^-} |\widehat{f}(t,\xi)|^2 dt d\xi\,,\ u \in \mathcal{D}(\omega \times I)$$

The proof of Theorem 4.1 is the same as that of Theorem 3.1, with the suitable changes, using (4.1) instead of (3.1).

Remark 4.2. It is easy to see that, if h is a C^1-local diffeomorphism near 0 with $h(0) = 0$, then ϵ-subellipticity for the system (1.3) is equivalent to ϵ-subellipticity for the system (1.3) with φ replaced by $\varphi \circ h$. So it is natural to ask : is the hypothesis (H_1) $\big($resp. (H_2) $\big)$ invariant under C^1- local diffeomorphisms h, $h(\circ) = 0$, near 0? It is the case:

Proposition 4.3. *Let h be a C^1-local diffeomorphism near 0 with $h(0) = 0$. If $\varphi \in C^1(\omega, \mathbb{R})$ satisfies (H_1) (resp. (H_2)), then $\varphi \circ h$ also satisfies (H_1) (resp. (H_2)).*

The proof of this proposition is direct, taking as subsets (ω_j) corresponding to $\varphi \circ h$, the inverse images by h of the subsets (ω_j) corresponding to φ and as mappings γ_j, the inverse images of the mappings γ_j corresponding to φ.

Now, we will give three classes of examples. The first classes are the simplest and one can use the result of Hörmander [3] to deduce subellipticity for these first classes. It is not the case for the others.

Recall again that if φ is in one of these classes, *i.e.*, satisfies (H_1) or (H_2), then $\varphi \circ h$ satisfies (H_1) or (H_2) for every C^1-local diffeomorphism near $0, h(0) = 0$.

5. First classes

5.1. Class $(b_1)_{p,\pm}$, $p \in \mathbb{N}^*$

It is the class of functions φ such that $\varphi(t) = \pm t_1^p + \widetilde{\varphi}(t_2, ., t_n)$, $\tilde{\varphi} \in C^1(V, \mathbb{R})$ with $0 \in V \subset \mathbb{R}^{n-1}$ and $\widetilde{\varphi}(0) = 0$.

Proposition 5.1.

1) *Let* $\varphi \in (b_1)_{p,+}$, $p \in \mathbb{N}^*$. *Then*
   a) *if* $p = 2q$, $q \in \mathbb{N}^*$: φ *satisfies* (H_1) *with* $\alpha = 2q$.
   b) *if* $p = 2q+1$, $q \in \mathbb{N}$: φ *satisfies* (H_1) *and* (H_2) *with* $\alpha = 2q+1$.
2) *Let* $\varphi \in (b_1)_{p,-}$. *Then*
   a) *if* $p = 2q, q \in \mathbb{N}^*$; φ *satisfies* $(H_2), \alpha = 2q$.
   b) *if* $p = 2q+1, q \in \mathbb{N}$; φ *satisfies* (H_1) *and* $(H_2), \alpha = 2q+1$.

Sketch of proof. Consider, say a small ball ω around $0 \in \mathbb{R}^n$. The subsets (ω_j) will be ω_1, ω_2, for the case 1) a) for example $\omega_1 = \{t \in \omega\,;\, t_1 \geq 0\}$; $\omega_2 = \{t \in \omega\,;\, t_1 < 0\}$.
So, here $\omega = \omega_1 \cup \omega_2$. Then we define γ_1 and γ_2 by:

$$\gamma_1(t,\tau) = (t_1 + \tau, t_2, ., t_n) \quad (t,\tau) \in \omega_1 \times [0,1],$$

$$\gamma_2(t,\tau) = (t_1 - \tau, t_2, .., t_n) \quad (t,\tau) \in \omega_2 \times [0,1].$$

In the case 1) b), we take all ω (*i.e.*, $k = 1$) and define the map γ on all ω by:

$$\gamma(t,\tau) = (t_1 + \tau, t_2, ., t_n)\,;\, (t,\tau) \in \omega \times [0,1]$$

Particularly in this last case 1) b), we use the following:

$$(a+\tau)^{2q+1} - a^{2q+1} \geq \left(\frac{1}{2}\right)^{2q} \tau^{2q+1}, \quad (a,\tau) \in \mathbb{R} \times [0,1].$$

in the proof of inequality (3.1) of (H_1).

To prove 2) a) and 2) b), we use similar $\omega_j\,'s$, and $\gamma_j\,'s$.

5.2. Class $(b_2)_{p,q,\pm}$, $p, q \in \mathbb{N}^*$

It is the class of functions φ such that $\varphi(t) = \pm\big[t_1^p + t_1^q\widetilde{\varphi}(t_2,.,t_n)\big]$; $p, q \in \mathbb{N}^*, \widetilde{\varphi} \in C^1(V, \mathbb{R}^+), 0 \in V \subset \mathbb{R}^{n-1}$.

Proposition 5.2.

1) *Let $\varphi \in (b_2)_{2p,2q,+}$; $p, q \in \mathbb{N}^*$. Then φ satisfies (H_1) with $\alpha = 2p$.*
2) *Let $\varphi \in (b_2)_{2p+1,2q,+1,\pm}$. Then φ satisfied (H_1) and (H_2) with $\alpha = 2p + 1$.*
3) *Let $\varphi \in (b_2)_{2p,2q,-}$. Then φ satisfies (H_2) with $\alpha = 2p$.*

For the proof of 1) for example, we use the same $\omega_j's$ and $\gamma_j's$ as in the case 1) a) of Proposition 5.1.

6. Homogeneous functions in case $n = 2$

Consider a real function φ, C^1 and homogeneous in $\mathbb{R}^2$: $\varphi(\lambda t) = \lambda^m\varphi(t), \lambda > 0$, $m \in \mathbb{N}^*, t \in \mathbb{R}^2$.

Denote by S the unit circle ; $\theta \in [-\pi, \pi[$ the variable on S. Let $\psi(\theta) = \varphi(\cos\theta, \sin\theta)$ $= \varphi(e^{i\theta})$. So we assume:

$(H)_\varphi$ $\begin{cases} \text{a) The function } \psi \text{ vanishes at a finite number of points } \theta_1, ., \theta_k \text{ of } S, \\ \text{where it changes sign. On the intervals }]\theta_j, \theta_{j+1}[\text{ where it is positive,} \\ \text{it admits only one local maximum and on the intervals where it is} \\ \text{negative, say }]\theta_l, \theta_{l+1}[, \text{ it admits only one local minimum.} \\ \text{b) There exist } c > 0 \text{ and } \epsilon > 0 \text{ such that } |\psi(\theta) - \psi(\theta')| \geq c\,|\theta - \theta'|^m, \\ \text{for } \theta, \theta' \in [\theta_j, -\epsilon, \theta_j, +\epsilon],\ j = 1, ., k. \end{cases}$

Proposition 6.1. *Let φ as above, satisfying $(H)_\varphi$. Then φ satisfies the hypotheses (H_1) and (H_2), with $\alpha = m$.*

Remark 6.2. In [1], the authors considered homogeneous polynomials and gave an (abstract) condition to obtain the maximal estimate (1.1) in the introduction.

Sketch of the proof. To show that φ satisfies (H_1), for example, we take as subsets ω_j of say $\omega = \{||t|| < 1/2\}$

$$\omega_j = \{t \in \omega, t = ||t||e^{i\theta},\ \theta \in]\,\theta_j, \theta_{j+1}\,[\,\},\ j = 1\ldots k$$

with $\theta_{k+1} = \theta_1$. For the choice of $\gamma_j's$, we consider two cases.

a) <u>The regions $\bar{\omega_j}$ where $\varphi \geq 0$</u>:

Call ω_j one of these regions and $a_1 \in]\,\theta_1, \theta_2\,[$ the point where ψ admits the local maximum (on $]\,\theta_1, \theta_2\,[\,)$. Then:

$$\gamma_1(t, \tau) = t + \tau\, e^{ia_1}\ ;\ (t, \tau) \in \bar{\omega_1} \times [0, 1].$$

b) The regions ω_l where $\varphi < 0$:
Here, for every $t \in \omega_2$, the curve γ_2, will have two C^1- pieces. So, call $b_2 \in]\,\theta_2, \theta_3\,[$ the point where ψ admits the (only) local minimum. Then:

$$\gamma_2(t,r) = t - \tau c^{ib_2}\ ,\ t \in\ [\,0, t_0\,]$$

with t_0 such that Arg $(t - t_0 e^{ib})$ equals θ_1 or θ_3 and

$$\begin{cases} \gamma_2(t,\tau) = t - t_0\, e^{ib_2} + (\tau - t_0)\, e^{ia_1} \text{if } \mathrm{Arg}(t - t_0\, e^{ib}) = \theta_1 \\ \gamma_2(t,\tau) = t - t_0\, e^{ib_2} + (\tau - t_0)\, e^{ia_3} \text{if } \mathrm{Arg}(t - t_0\, e^{ib}) = \theta_3 \\ \tau \in [t_0, 1]. \end{cases}$$

Corollary 6.3. *Let $n > 2$ and $\varphi = \varphi_1 + \varphi_2$ where $\varphi_1(t) = \varphi_1(t_1, t_2)$ as in Proposition 6.1 and $\varphi_2 = \varphi_2(t_3, ., t_n)$ a C^1 real function defined on a neighborhood of 0 in $\mathbb{R}^{n-2}$ then φ satisfies (H_1) and (H_2).*

One has just to take the maps γ_j defined in the preceding sketch of proof of Proposition 6.1, independent of the variables $(t_3, ., t_n)$.

Explicit example:
$\varphi(t) = t_1(t_1^{2l} - t_2^{2l}) + \widetilde{\varphi}(t_3, ., t_n)$ with $l \in \mathbb{N}^*$ $\widetilde{\varphi}(0) = 0$, $\widetilde{\varphi} \in C^1(V, \mathbb{R})$, $0 \in V \subset \mathbb{R}^{n-2}$. In that case, one has just to study the function $\Psi(\theta) = \cos\theta\,(\cos^{2l}\theta - \sin^{2l}\theta)$ on $[\,0, \pi\,]$.

7. The examples of H. Maire

These are the functions φ defined (on $\mathbb{R}^2$) by $\varphi(t) = t_1(t_1^{2l} - t_2^2)$ for $l \in \mathbb{N}^*$.

If $l = 1, \varphi$ is homogeneous and is considered before. If $l > 1$, then one can split a small ball around 0, into a finite number of regions where one can give maps γ_j in order to satisfy (H_1) (resp (H_2)).

Remark 7.1. In fact, the example of H. Maire is a special case of quasihomogeneous functions for which we have a result analogous to the one given for the homogeneous functions.

Final remarks

1. In the case φ is real analytic it is believed that the non existence of a local minimum of φ in a neighborhood of 0 implies the microlocal subellipticity in the positive direction, and the non existence of a local maximum implies microlocal subellipticity in the negative direction as suggested to me by H. Maire. I hope that this may be done via (H_1) (resp (H_2)).
2. A recent result of J.J. Kohn [4] shows that, in the case of complex vector fields, the inequality (1.2) is valid with only a “very negative” epsilon (negative with great absolute value, as great as one want).

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Approximate Solutions to the 2-D Unsteady Navier-Stokes System with Free Surface

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Abstract. In this note we present a method based on Galerkin scheme that seems appropriate to provide global in time fluids flows in domains with moving boundary. Initial data are assumed to be small but not infinitesimal.

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1. Introduction

One of the main difficulties in the mathematical approach to the study of fluid flows occurring in domains with free surface, arises from the fact that unknown velocity field must be determined in an unknown domain. Usually existence proofs of unsteady solutions employ an approximating scheme, where at each step the approximating domain is reduced to a fixed one, we quote for instance [2] and [5] and the references therein. Such method allows for solutions which may exist globally in time, only for very small, not controllable initial data. Our aim is to present an approximating scheme, based on Galerkin procedure, that may provide global in time solutions for the free boundary problem with small, controllable initial data. Since at the same step we compute both the velocity field and the unknown domain, this method, introduced in [4] to study a model describing the interaction of a fluid with an elastic vessel, seems appropriate also for numerical simulations.

Let us now introduce the problem formulation.

Suppose that at each moment $t \in [0,T]$ the fluid occupies a two dimensional layer $\Omega_\eta(t) = \{(x,y) : x \in (0,1),\ 0 < y < 1 + \eta(x,t)\} \subset \mathbb{R}^2$, with rigid bottom $y = 0$, and upper surface $\Gamma_\eta := \{(x,y) : y = 1 + \eta(x,t),\ x \in (0,1),\ t \in (0,T)\}$. We will

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assume that $\mathrm{meas}(\Omega_\eta)$ is constant. The velocity field $\mathbf{v}$ defined in the curvilinear slab $G_\eta = \bigcup_{t\in(0,T)} \Omega_\eta(t)\times\{t\}$, $\mathbf{v} : G_\eta \mapsto \mathbb{R}^2$, satisfies the Navier-Stokes equations, namely

$$\begin{aligned} \nabla\cdot\mathbf{v} &= 0, && (x,y,t)\in G_\eta;\\ \partial_t\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v} &= \nu\nabla\cdot\mathbf{S}(\mathbf{v}) - \nabla p, && (x,y,t)\in G_\eta, \end{aligned} \tag{1.1a}$$

where ν is the kinematical viscosity, $\mathbf{S}(\mathbf{v})$ denotes the symmetric tensor $\mathbf{S}(\mathbf{v}) = \nabla\mathbf{v} + \nabla\mathbf{v}^T$. The following kinematical condition links the velocity field to the free boundary and expresses the fact the fluid particles cannot leave the fluid region G_η,

$$\partial_t\eta = -\partial_x\eta v_1 + v_2 = (\mathbf{v}\cdot\tilde{\mathbf{n}}), \qquad \text{on } \Gamma_\eta(t), \tag{1.1b}$$

where v_1 and v_2 are the horizontal and the vertical component of the velocity $\mathbf{v}$. Setting $g(\eta) = 1 + (\partial_x\eta)^2$ then $\mathbf{n} = (n_x, n_y) = (-\partial_x\eta, 1)/g(\eta)^{1/2}$ is the outward normal to Γ_η and we define $\tilde{\mathbf{n}}(\eta) = \sqrt{g(\eta)}\mathbf{n}(\eta) = (-\partial_x\eta(x,t), 1)$.

Concerning the boundary conditions we assume adherence of the fluid at the bottom of the layer and that the stress difference is proportional to the mean curvature at the free surface. This leads to the following constitutive equations that describe the dynamical behavior of the free boundary

$$\begin{gathered} \big(-p + \nu\mathbf{n}\cdot\mathbf{S}(\mathbf{v})\cdot\mathbf{n}\big)(x,\eta,t) = \sigma\partial_x\Big(\frac{\partial_x\eta}{\sqrt{g(\eta)}}\Big)(x,t) = H(\eta);\\ \big(\mathbf{t}\cdot\mathbf{S}(\mathbf{v})\cdot\mathbf{n}\big)(x,\eta,t) = 0\,,\\ \mathbf{v}(x,0,t) = 0, \quad (x,t)\in(0,1)\times[0,T). \end{gathered} \tag{1.1c}$$

Here the positive coefficient σ represents the surface tension. Finally we assume the following initial conditions

$$\begin{gathered} \mathbf{v}(x,y,0) = \mathbf{v}_0(x,y), \quad x,y\in\Omega_{\eta_0};\\ \eta(x,0) = \eta_0(x), \quad x\in(0,1), \quad \int_0^1 \eta_0(x)dx = 0. \end{gathered}$$

Moreover $\eta(x,t)$, $\mathbf{v}(x,y,t)$ assumed to be $1-$periodic with respect to the x axis.

This model formulation presented above is consistent if the following two conditions hold

(C1) the free boundary does not touch the rigid bottom $y = 0$;

(C2) $\sup_{(x,t)} |\partial_x\eta(x,t)| < +\infty$, namely for every instant of time and every point in $(0,1)$ the function describing the shape of the free boundary has no vertical tangent points.

Here we show that under suitable smallness assumption on initial data the first condition cannot occur. If the second condition fails to hold then a reversal flow may take place. We refer to the last section of this note for a discussion on this issue.

From now on the subscript $\sharp$ below capital letters denotes we are considering functional spaces consisting of $1-$periodic functions with respect to the x axis. Therefore we denote by $L^r_\sharp(\Omega_\eta(t))$ and $H^s_\sharp(\Omega_\eta(t))$, the Banach spaces which consist of all functions $u : \Omega_\eta(t) \mapsto \mathbb{R}$ having finite norms $\|\cdot\|_{L^r(\Omega_\eta(t))}$ and $\|\cdot\|_{H^s(\Omega_\eta(t))}$

respectively. Notations $L^r_\sharp$, $H^s_\sharp$, $W^{1,\infty}_\sharp$, $C^l_\sharp$ stand for Banach spaces of all functions $\eta : (0,1) \mapsto \mathbb{R}$ supplemented with the norms $\|\cdot\|_{L^r(0,1)}$, $\|\cdot\|_{H^s(0,1)}$, $\|\cdot\|_{W^{1,\infty}(0,1)}$, $\|\cdot\|_{C^l(0,1)}$.

Setting $\Omega = [0,1[\times[0,2[$, $C^\infty_{0,\sharp}(\Omega)$ stands for the set of smooth functions Ψ, with supp $\Psi \subset \Omega$ and $H^2_{0,\sharp}(\Omega)$ stands for its closure with respect to the $H^2_\sharp(\Omega)$-norm. Finally we denote by $\mathcal{V}^1$ the closed Hilbert subspace of $(H^1_\sharp(\Omega))^2$, which consists of all divergence-free vector fields.

Consider now the following eigenvalues problem

$$\Delta^2\psi^k = \lambda_k\psi^k \text{ in } \Omega, \quad \psi^k \in H^2_{0,\sharp}(\Omega), \quad \|\psi^k\|_{L^2(\Omega)} = 1. \tag{1.2}$$

Then $\{\psi^k\}_{k\in\mathbb{N}}$ form an orthogonal basis in $H^2_{0,\sharp}(\Omega)$ and clearly the vector fields

$$\mathbf{a}_k(x,y) = \text{sgrad } \psi^k := (\partial_y\psi^k, -\partial_x\psi^k) \tag{1.3}$$

form a basis in $\mathcal{V}^1$.

As said at the beginning, our approach in solving problem (1.1) relies on the Galerkin scheme exploited in a double approximating procedure, one regarding the velocity equation, the other the shape of the free surface. The key point in such a procedure is the observation that, for any function η that does not touch the bottom $y = 0$, the vector fields $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$, that form a basis of $\mathcal{V}^1$, generate the whole space of divergence free vectors belonging to $H^1_\sharp(\Omega_\eta)$. Moreover, thanks to their analyticity property, any finite number of such vector fields is linearly independent in $H^1_\sharp(\Omega_\eta)$.

This note is organized as follows: at first we construct the approximating solution to problem (1.1) essentially projecting both the momentum equation and the kinematical one's into finite dimensional spaces. Therefore system (1.1) is reduced to solving a Cauchy problem, which is not in normal form, since $\{\mathbf{a}_k\}_{k\in\mathbb{N}}$ are orthonormal on Ω but not on Ω_η. Here the smallness of initial data together with an energy estimate allows us to reduce such a differential system in normal form, thus obtaining a local in time approximate solution to problem (1.1). Then again the energy estimate ensures that this solution is also global in time. We finally quote [6] and [1] for other existence theorems of global in time solutions in the case when the free surface is not represented in cartesian coordinates.

2. Approximate solutions

Assume that initial data $\eta_0 \in W^{1,\infty}_\sharp$ and $\mathbf{v}_0 \in L^2_\sharp(\Omega_{\eta_0})$ are such that

$$\eta_0 > -1, \quad \text{div}\,\mathbf{v}_0 = 0. \tag{2.1}$$

The regularity of η_0 allows us to extend $\mathbf{v}_0$ to the whole domain Ω retaining its divergence free property. Let us still denote by $\mathbf{v}_0$ such an extension. We are now in a position to define an approximate solution to problem (1.1). Consider the functions $\mathbf{v}^{mn}(x,y,t)$, $\eta^{mn}(x,t)$ having the representation

$$\mathbf{v}^{mn}(x,y,t) = \sum_{p=1}^{n} c_p^{mn}(t)\mathbf{a}_p(x,y), \quad \eta^{mn}(x,t) = \sum_{k=-m}^{m} f_k^{mn}(t)e^{-i2\pi kx}. \tag{2.2}$$

Given $u \in H^1_\sharp(0,1)$ let us introduce the projector Π_m defined by

$$\Pi_m u = \sum_{k=-m}^{m} u_k e^{-i2\pi kx}, \quad u_k = \int_0^1 u(x)e^{2\pi ikx}\,dx.$$

Definition 2.1. The couple $(\eta^{mn}, \mathbf{v}^{mn})$ is an approximate solution to problem (1.1) if

(i) η^{mn} and $\mathbf{v}^{mn}$ have representation (2.2) with c_p^{mn}, $f_k^{mn} \in C^1(0,T)$ and

$$0 < 1 + \eta^{mn}(x,t) < 2 \quad \text{for all } (x,t) \in (0,1)\times[0,T) \tag{2.3}$$

(ii) For all smooth $\varphi = e_l(t)\mathbf{a}_l$, $1 \le l \le n$, vanishing at $t = T$,

$$\begin{aligned} &\int_0^T \int_{\Omega_{\eta^{mn}}(t)} \Big(\mathbf{v}^{mn}\cdot\partial_t\varphi - \frac{\nu}{2}S(\mathbf{v}^{mn}) : S(\varphi) + \mathbf{v}^{mn}\otimes\mathbf{v}^{mn} : \nabla\varphi\Big)\,dx\,dy\,dt \\ &+\frac{1}{2}\int_0^T\int_0^1 \Big\{\Big(\partial_t\eta^{mn} - \mathbf{v}^{mn}\cdot\tilde{\mathbf{n}}\Big)\mathbf{v}^{mn}\cdot\varphi\Big\}_{y=1+\eta^{mn}}\,dx\,dt \\ &+\int_{\Omega_{\eta^{mn}}(0)} \mathbf{v}_0\cdot\varphi(x,y,0)\,dx\,dy + \sigma\int_0^T\int_0^1 \mathcal{H}(\eta^{mn})(\varphi\cdot\tilde{\mathbf{n}}^{mn})_{y=1+\eta^{mn}}\,dx\,dt = 0, \end{aligned} \tag{2.4}$$

where $\tilde{\mathbf{n}}^{mn} = \tilde{\mathbf{n}}(\eta^{mn})$ and $\mathcal{H}(\eta^{mn}) = \Pi_m\partial_x\Big(\frac{\partial_x\eta^{mn}}{\sqrt{g^{mn}}}\Big)$.

(iii) The equation and initial condition

$$\partial_t\eta^{mn}(x,t) = \Pi_m(\mathbf{v}^{mn}\cdot\tilde{\mathbf{n}}^{mn})_{y=1+\eta^{mn}}(x,t), \quad \eta^{mn}(\cdot,0) = \Pi_m\eta_0 \tag{2.5}$$

hold in $(0,1)\times[0,T)$.

(iv) Functions $(\eta^{mn}, \mathbf{v}^{mn})$ satisfy the energy inequality

$$\begin{aligned} &\sup_{(0,T)}\Big\{\sigma\|\sqrt{1+(\partial_x\eta^{mn})^2(t)}\|_{L^1((0,1)} + \frac{1}{2}\|\mathbf{v}^{mn}(t)\|^2_{L^2(\Omega_{\eta^{mn}}(t))}\Big\} \\ &\qquad + \frac{\nu}{2}\int_0^T\int_{\Omega_{\eta^{mn}}(t)} |S(\mathbf{v}^{mn})|^2\,dx\,dy\,dt \le cE_0, \end{aligned} \tag{2.6}$$

for some $c > 1$, where

$$E_0 := \frac{1}{2} \int\limits_{\Omega_{\eta_0}} |\mathbf{v}_0|^2 \, dx \, dy + \sigma \int\limits_0^1 \sqrt{1 + (\partial_x \eta_0)^2} \, dx.$$

From now on, if no confusion arises, we omit the subscript $y = 1 + \eta^{mn}$.

We can now state the main result of this paper.

Theorem 2.2. *Fixed $\delta \in (0, \sqrt{2} - 1)$ suppose that initial data have finite energy E_0 such that*

$$E_0 < \sigma(1 + \delta). \tag{2.7}$$

Then there is $M > 0$ such that for all $m \geq M$ and $n \geq 1$, problem (1.1) has a global in time approximate solution.

Remark 2.3. This theorem tells us that conditions (C1) cannot occur when passing to the limit in the approximating scheme.

Normal form of the Galerkin equations. Set

$$\varphi = e(t)\mathbf{a}_l(x, y), \quad e(T) = 0\,,$$

with $e(t)$ smooth. Recalling (2.2), integrating by parts (2.4) with respect to time and noting that $\partial_t \eta^{mn} = \Pi_m(\mathbf{v}^{mn} \tilde{\mathbf{n}}^{mn})$, we get

$$\begin{aligned}
&\int\limits_0^T e(t) \int\limits_{\Omega_{\eta^{mn}}(t)} \Big(-\frac{d}{dt} c_p^{mn}(t) \mathbf{a}_p \cdot \mathbf{a}_l - c_p^{mn}(t) \frac{\nu}{2} S(\mathbf{a}_p) : S(\mathbf{a}_l) \Big) dx \, dy \, dt \\
&+ \int\limits_0^T e(t) \int\limits_{\Omega_{\eta^{mn}}(t)} c_p^{mn}(t) c_q^{mn}(t) \mathbf{a}_p \otimes \mathbf{a}_q : \nabla \mathbf{a}_l \, dx \, dy \, dt \\
&- \int\limits_0^T e(t) \int\limits_0^1 \frac{1}{2} c_p^{mn}(t) c_q^{mn}(t) (\Pi_m(\mathbf{a}_p \cdot \tilde{\mathbf{n}}^{mn}) + \mathbf{a}_p \cdot \tilde{\mathbf{n}}^{mn})(\mathbf{a}_q \cdot \mathbf{a}_l) \, dx \, dt \\
&- \int\limits_0^T e(t) \int\limits_0^1 \sigma \mathcal{H}(\eta^{mn}) \mathbf{a}_l \cdot \tilde{\mathbf{n}}^{mn} \, dx \, dt + e(0) \int\limits_{\Omega_{\eta^{mn}}(0)} \mathbf{v}_0 \cdot \mathbf{a}_l \, dx \, dy \\
&- e(0) \int\limits_{\Omega_{\eta^{mn}}(0)} c_p^{mn}(0) \mathbf{a}_p \cdot \mathbf{a}_l \, dx \, dy = \int\limits_0^T I_t \, dt + J_0 = 0\,.
\end{aligned} \tag{2.8}$$

Since $e(t)$ is arbitrary from the previous system we deduce that both $I_t = 0$ and $J_0 = 0$. Now set

$$a_{lp}^{mn}(\mathbf{f}) = \int\limits_{\Omega_{\eta^{mn}}(t)} \mathbf{a}_p \cdot \mathbf{a}_l \, dx \, dy, \qquad b_{lp}^{mn}(\mathbf{f}) = -\frac{\nu}{2} \int\limits_{\Omega_{\eta^{mn}}(t)} S(\mathbf{a}_p) : S(\mathbf{a}_l) \, dx \, dy, \tag{2.9a}$$

$$v_{lpq}^{mn}(\mathbf{f}) = \int\limits_{\Omega_{\eta^{mn}}(t)} \mathbf{a}_p \otimes \mathbf{a}_q : \nabla \mathbf{a}_l \, dx \, dy, \tag{2.9b}$$

$$s_{lpq}^{mn}(\mathbf{f}) = -\frac{1}{2} \int\limits_0^1 (\Pi_m \mathbf{a}_p \cdot \tilde{\mathbf{n}}^{mn} + \mathbf{a}_p \cdot \tilde{\mathbf{n}}^{mn})(\mathbf{a}_q \cdot \mathbf{a}_l) \, dx, \tag{2.9c}$$

$$l(\mathbf{f})_l^{mn} = -\sigma \int\limits_0^1 \mathcal{H}(\eta^{mn}) \mathbf{a}_l \cdot \tilde{\mathbf{n}}^{mn} \, dx \, dy, \qquad \mathbf{c}_{0,l}^{mn} = \int\limits_{\Omega_{\eta^{mn}}(0)} \mathbf{v}_0 \cdot \mathbf{a}_l(x, 0) \, dx \, dy \,, \tag{2.9d}$$

$$g_{kp}^{mn}(\mathbf{f}) = \int\limits_0^1 (\mathbf{a}_p \cdot \tilde{\mathbf{n}}^{mn}) e^{i2\pi kx} \, dx, \qquad -m \le k \le m. \tag{2.9e}$$

If we denote by $\mathbf{c}(t) = (c_p^{mn}(t))_{1 \le p \le n}$, $\mathbf{f}(t) = (f_k(t))_{-m \le k \le m}$, and by

$$\mathcal{A}(\mathbf{f}), \quad \mathcal{B}(\mathbf{f}), \quad \mathcal{V}(\mathbf{f}), \quad \mathcal{S}(\mathbf{f}), \quad \mathbf{l}(\mathbf{f}), \quad \mathbf{c}_0, \quad \mathcal{G}(\mathbf{f}),$$

the matrices whose coefficients are given by (2.9), then (2.8) and (2.5) are equivalent to the following system

$$\mathcal{A}(\mathbf{f}) \frac{d\mathbf{c}}{dt} = \mathcal{B}(\mathbf{f}) \mathbf{c} + \mathcal{D}(\mathbf{f})[\mathbf{c}, \mathbf{c}] + \mathbf{l}(\mathbf{f}) \tag{2.10a}$$

$$\frac{d\mathbf{f}}{dt} = \mathcal{G}(\mathbf{f}) \mathbf{c} \tag{2.10b}$$

$$\mathcal{A}(\mathbf{f}_0) \mathbf{c}(0) = \mathbf{c}_0, \quad \mathbf{f}(0) = \mathbf{f}_0. \tag{2.10c}$$

where $\mathcal{D}(\mathbf{f}) = \mathcal{V}(\mathbf{f}) + \mathcal{S}(\mathbf{f})$ and

$$\mathcal{D}(\mathbf{f})[\mathbf{c}, \mathbf{c}] = \sum_{p,q=1}^{n} d_{lpq}^{mn}(\mathbf{f}) c_p^{mn} c_q^{mn} = \sum_{p,q=1}^{n} (v_{lpq}^{mn} + s_{lpq}^{mn})(\mathbf{f}) c_p^{mn} c_q^{mn}.$$

Equations (2.10) form a system of first order ordinary differential equations and we wish to put it in normal form. We achieve this goal once we show that the coefficients of these equations are smooth functions of $\mathbf{f}$ and the matrix $\mathcal{A}$ is invertible. Set

$$\varepsilon = \sqrt{2} - 1 - \delta > 0 \tag{2.11}$$

and denote by $\mathbb{K} \subset \mathbb{R}^{2m}$ the bounded, open, convex subset of $\mathbb{R}^{2m}$ which consists of all $\mathbf{f}$ satisfying the inequalities

$$\varepsilon < 1 + \eta^{mn}(x, t) < 2 - \varepsilon, \qquad \text{for all } x \in [0, 1). \tag{2.12}$$

Lemma 2.4. *For each $\alpha > 0$,*

$$a_{lp}^{mn},\, b_{lp}^{mn},\, d_{lpq}^{mn},\, g_{kp}^{mn} \in C^\alpha(\mathbb{K}), \quad \mathbf{e}^{mn} \in C^\alpha(\mathbb{K})^n, \quad \mathcal{A}^{-1} \in C^\alpha(\mathbb{K})^{n^2}. \tag{2.13}$$

Proof. Let us note that η^{mn} and components of the vector $\tilde{\mathbf{n}}^{mn}$ are affine function of f_k with analytic coefficients. Hence inclusions (2.13) hold true for $\mathcal{S}$, $\mathcal{G}$ and the vector-valued function $\mathbf{e}$. It is easy to see that for every k

$$\partial_{f_k} a_{lp}^{mn}(\mathbf{f}) = \int_0^1 (\mathbf{a}_p \cdot \mathbf{a}_l) e^{i2\pi kx}\, dx\,.$$

Hence $\mathcal{A} \in C^\alpha(\mathbb{K})^{n^2}$ for all positive α; the same conclusion can be drawn for $\mathcal{B}$ and $\mathcal{V}$. It remains to prove the smoothness of the matrix $\mathcal{A}^{-1}$. It suffices to show that for all $\mathbf{f} \in \mathbb{K}$,

$$\|\mathcal{A}(\mathbf{f})^{-1}\| \le c(\varepsilon, n). \tag{2.14}$$

Note that

$$\|\mathcal{A}(\mathbf{f})^{-1}\| = \Big\{ \inf_{|\lambda|=1} \big(\mathcal{A}(\mathbf{f})\lambda \cdot \lambda\big) \Big\}^{-1} \text{ and } \mathcal{A}(\mathbf{f})\lambda \cdot \lambda = \int_{\Omega_{\eta^{mn}}} \Big| \sum_{p=1}^n \lambda_p \mathbf{a}_p \Big|^2 dx\, dy$$

where η^{mn} are related to $\mathbf{f}$ by relation (2.2). Since the vector fields $\mathbf{a}_p$ are analytic and linearly independent in Ω, they are linearly independent on each open connected subset of Ω, which along with the definition of $\mathbb{K}$ yields

$$\int_{\Omega_{\eta^{mn}}} \Big| \sum_{p=1}^n \lambda_p \mathbf{a}_p \Big|^2 dx\, dy \ge \int_0^1 \int_0^\varepsilon \Big| \sum_{p=1}^n \lambda_p \mathbf{a}_p \Big|^2 dx\, dy \ge c(\varepsilon, n)^{-1} |\lambda|^2 > 0 \tag{2.15}$$

for every $\lambda \in \mathbb{R}$, and the lemma follows. □

Energy estimate. Here we derive the energy estimate for (2.10). Suppose that an approximate solution to problem (1.1) is defined on the interval $(0, \tau) \subset (0, T)$. Set

$$\zeta(t) = 1 \text{ for } t \in [0, \tau - \epsilon), \quad \zeta(t) = \frac{\tau - t}{\epsilon} \text{ for } t \in [\tau - \epsilon, \tau) \text{ and } \zeta(t) = 0 \text{ for } t > \tau.$$

Substituting $\varphi = \zeta(t)\mathbf{v}^{mn}(x, y, t)$ into the integral identity (2.4) we arrive at

$$\begin{aligned}
&- \int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} \Big(|\mathbf{v}^{mn}|^2 \partial_t \zeta + \frac{1}{2} \partial_t |\mathbf{v}^{mn}|^2 \zeta + \zeta \mathbf{v}^{mn} \otimes \mathbf{v}^{mn} : \nabla \mathbf{v}^{mn} \Big)\, dx\, dy\, dt \\
&+ \frac{\nu}{2} \int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} \zeta |S(\mathbf{v}^{mn})|^2\, dx\, dy\, dt - \sigma \int_0^\tau \int_0^1 \zeta \mathcal{H}(\eta^{mn})(\mathbf{v}^{mn} \cdot \tilde{\mathbf{n}}^{mn})\, dx\, dt \\
&- \frac{1}{2} \int_0^T \int_0^1 \zeta(t) \Big(\partial_t \eta^{mn} - \mathbf{v}^{mn} \cdot \tilde{\mathbf{n}}^{mn} \Big) |\mathbf{v}^{mn}|^2\, dx\, dt = \int_{\Omega_{\eta^{mn}}(0)} \mathbf{v}_0 \cdot \mathbf{v}^{mn}(0)\, dx\, dy\,.
\end{aligned} \tag{2.16}$$

Since $\partial_t \eta^{mn}(x,0) = \Pi_m\Big(\mathbf{v}^{mn}(x,\eta^{mn},0)\tilde{\mathbf{n}}^{mn}(x,0)\Big)$ then

$$-\int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} \partial_t |\mathbf{v}^{mn}|^2 \zeta \, dx\, dy\, dt = \int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} |\mathbf{v}^{mn}|^2 \partial_t \zeta \, dx\, dy\, dt$$

$$+ \int_{\Omega_{\eta^{mn}}(0)} |\mathbf{v}^{mn}(x,y,0)|^2 dxdy + \int_0^\tau \int_0^1 \zeta |\mathbf{v}^{mn}|^2 \, \partial_t \eta^{mn} \, dx\, dt,$$

$$\int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} \zeta \mathbf{v}^{mn} \otimes \mathbf{v}^{mn} : \nabla \mathbf{v}^{mn} dx\, dy\, dt = \frac{1}{2} \int_0^\tau \int_0^1 \zeta \Big(|\mathbf{v}^{mn}|^2 \mathbf{v}^{mn} \cdot \tilde{\mathbf{n}}^{mn}\Big) \, dx\, dt\,.$$

Using these identities and passing to the limit with respect to ϵ in (2.16) we get

$$\frac{1}{2} \int_{\Omega_{\eta^{mn}}(\tau)} |\mathbf{v}^{mn}(x,y,\tau)|^2 \, dx\, dy + \frac{\nu}{2} \int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} |S(\mathbf{v}^{mn})|^2 \, dx\, dy\, dt$$

$$- \sigma \int_0^\tau \int_0^1 \mathcal{H}(\eta^{mn})(\mathbf{v}^{mn} \cdot \tilde{\mathbf{n}}^{mn}) dt dx = \int_{\Omega_{\eta^{mn}}(0)} \mathbf{v}_0 \cdot \mathbf{v}^{mn}(x,y,0) \, dx\, dy$$

$$- \int_{\Omega_{\eta^{mn}}(0)} \frac{1}{2} |\mathbf{v}^{mn}(x,y,0)|^2 \, dx\, dy \le \frac{1}{2} \int_{\Omega_{\eta^{mn}}(0)} |\mathbf{v}_0|^2 \, dx.$$

As $\Pi_m \eta^{mn} = \eta^{mn}$ and $\Pi_m \mathcal{H}(\eta^{mn}) = \mathcal{H}(\eta^{mn})$, integration by parts yields

$$-\int_0^\tau \int_0^1 \mathcal{H}(\eta^{mn})(\mathbf{v}^{mn} \cdot \tilde{\mathbf{n}}^{mn}) dt dx = \int_0^1 g^{mn}(\tau)^{1/2} \, dx - \int_0^1 g^{mn}(0)^{1/2} \, dx.$$

Thus we finally have

$$\frac{1}{2} \int_{\Omega_{\eta^{mn}}(\tau)} |\mathbf{v}^{mn}(\tau)|^2 \, dx\, dy + \sigma \int_0^1 g^{mn}(\tau)^{1/2} \, dx + \frac{\nu}{2} \int_0^\tau \int_{\Omega_{\eta^{mn}}(t)} |S(\mathbf{v}^{mn})|^2 \, dx\, dy\, dt$$

$$\le \frac{1}{2} \int_{\Omega_{\eta^{mn}}(0)} |\mathbf{v}_0|^2 \, dx\, dy + \sigma \int_0^1 g^{mn}(0)^{1/2} \, dx =: E_m. \tag{2.17}$$

Proof of Theorem 2.2. Let us show that there exists $M > 0$ such that for any $m > M$ $E_m \le cE_0$ with $c > 1$. Recall that by assumption we have in particular that $\eta_0 \in H^1_\sharp$, so that

$$\eta^{mn}(\cdot,0) = \Pi_m \eta_0 \to \eta_0 \text{ in } C^0_\sharp, \quad \partial_x \eta^{mn}(\cdot,0) = \Pi_m \partial_x \eta_0 \to \partial_x \eta_0 \text{ in } L^2_\sharp \text{ as } m \to \infty.$$

Since $\mathbf{v}_0 \in L^2(\Omega)$, the previous convergence properties imply that

$$\frac{1}{2}\Big(\int\limits_{\Omega_{\Pi_m \eta_0}} - \int\limits_{\Omega_{\eta_0}} \Big) |v_0|^2 \, dx \, dy + \sigma \int\limits_0^1 \sqrt{g^{mn}(0)} - \sqrt{g(0)} \, dx \to 0 \text{ as } m \to \infty. \tag{2.18}$$

Hence there exists $M(\eta_0, \mathbf{v}_0) > 0$ such that for all $m \geq M$,

$$|\eta_0(x) - \eta^{mn}(x,0)| < \varepsilon/4, \quad |E_m - E_0| < \sigma\varepsilon/4, \quad E_m \leq cE_0 \tag{2.19}$$

for some $c > 1$. In order to apply Cauchy theorem to solve system (2.10) we have first to show that initial data belong to $\mathbb{K}$. To this aim notice that

$$\eta_0(x) = \int_0^1 \left(\eta_0(y) - \int_x^y \partial_x \eta_0(z) \, dz \right) dy \, .$$

Since by hypothesis η_0 has zero mean value in the interval (0,1), by (2.7) we have

$$\begin{aligned} 1 + |\eta_0(x)| &\leq \int\limits_0^1 (1 + |\partial_x \eta_0(x)|) \, dx' \leq \sqrt{2} \int\limits_0^1 \sqrt{1 + (\partial_x \eta_0(x))^2} \, dx \leq \sqrt{2} \frac{E_0}{\sigma} \\ &\leq \sqrt{2}(1 + \delta) \, , \end{aligned}$$

therefore recalling (2.11) we have

$$0 < \varepsilon < 2 - \sqrt{2}(1+\delta) \leq 1 + \eta_0(x) \leq \sqrt{2}(1+\delta) < 2 - \varepsilon \, . \tag{2.20}$$

Hence inequalities (2.19) lead to the estimate

$$\varepsilon < 1 + \eta^{mn}(x,0) < 2 - \varepsilon \, ,$$

which yields $\mathbf{f}_0 \in \mathbb{K}$. From this, Lemma 2.4 and the Cauchy Theorem we conclude that problem (2.10) has a unique solution $(\mathbf{c}(t), \mathbf{f}(t)) \in \mathbb{R}^n \times \mathbb{K}$ defined on the interval $[0, \tau)$ for some $\tau \in (0, T]$. Let τ^* be the upper bound for all such τ. Theorem 2.2 will be proved once we show that $\tau^* = T$ for each $T > 0$.

By energy estimate (2.17), a solution to problem (2.10) is bounded and continuous on $[0, \tau^*]$. If we assume by contradiction that $\tau^* < T$, then clearly $\mathbf{f}(\tau^*) \in \partial\mathbb{K}$. Arguing in the same way we did to prove (2.20) by the energy estimate (2.17) and (2.19) we deduce that

$$1 + |\eta^{mn}(x,t)| \leq \sqrt{2} \frac{E_m}{\sigma} < \sqrt{2} \Big(\frac{E_0}{\sigma} + \frac{\varepsilon}{4} \Big) \, .$$

Taking into account (2.11) and (2.7), then also η^{mn} satisfies

$$\varepsilon < 1 + \eta^{mn}(x,t) < 2 - \varepsilon \qquad \text{for all } (x,t) \in [0,1) \times [0, \tau^*] \, ,$$

therefore $\mathbf{f}(\tau^*) \in \text{int } \mathbb{K}$, in contradiction with the inclusion $\mathbf{f}(\tau^*) \in \partial\mathbb{K}$, and the theorem follows. $\square$

3. Final remarks

1) *Passage to the limit.* Here we do not analyze the passage to the limit, nevertheless we observe that it can be done, following the proof contained in [4], if the boundary is regular enough. The first step consists in proving a compactness principle that yields the convergence of the velocity field in a space regular enough to deal with the nonlinear convective term in the Navier-Stokes equations. Since for any $n \in \mathbb{N}$ the boundary is analytic with respect to the x variable, the passage to the limit with respect to the index n is simple. Solving suitable Dirichlet problems with moving boundaries and exploiting again the strong convergence of the velocity field it is possible to pass to the limit also with respect to m, see [4]. We stem that in order to get a solution to problem (1.1), the regularity of the approximating solutions, that can be obtained by the energy estimate, is crucial.

2) *Reversal flow.* Notice that the second component of the outward normal to the free surface of the approximating scheme is given by

$$j^{mn}(x,t) = \frac{1}{\sqrt{1+(\partial_x \eta^{mn}(x,t))^2}} > 0 \qquad \text{for all } (x,t) \in (0,1) \times [0,T)$$

and this bound holds for every n and m. On the other hand the energy estimate (2.17) implies that

$$\sup_{[0,T]} \int_0^1 \sqrt{1+(\partial_x \eta^{mn})^2}\, dx < +\infty$$

hence when passing to the limit with respect to m (the "free boundary" index) there may exist $x \in (0,1)$ and $t \in [0,T)$ such that the limiting function $j(x,t) = 0$, thus condition (C2) is violated.

Observe that the occurrence of such a phenomenon is only due to the horizontal component of the velocity field $\mathbf{v}$. Indeed if we assume that $v_1(x, 1+\eta, t) = 0$ for any x and t then condition (C2) cannot occur. This hypothesis has been employed [3] to study an fluid-elastic structure interaction model. We point out that, from a physical point of view, this assumption, though suitable to describe porous flows, is very restrictive for the free boundary case. We wonder weather sufficient conditions to avoid a reversal flow can be obtained by the constitutive equation of the free boundary, see (1.1c).

3) *Extensions.* The method exposed here works in general if, instead of the mean curvature, we take into account other operators H in (1.1c) with sufficiently large order. If for instance we consider $H = -\partial_x^4 \eta$ by the energy estimate the approximate solution to problem (1.1) would satisfy

$$\sup_{[0,T]} \|\partial_x^2 \eta^{mn}\|_{L^2(0,1)}(t) \le c(E_0)$$

that hinders the occurrence of reversal flow. It goes without saying that the free boundary will not touch the bottom provided E_0 is sufficiently small. We deem also that the smallness hypothesis E_0 may be dropped by taking an infinite layer. Existence of three dimensional flows may be studied in a similar way by adding more regularity on the operator H. Another challenging problem consists in considering a rigid container with contact line at the intersection with the free boundary.

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Time Decay Estimates of Solutions for Wave Equations with Variable Coefficients

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Abstract. The aim of this work is to derive the time decay estimates of solutions for wave equations in $\mathbb{R}^n$ with variable coefficients which are constants near the infinity of $\mathbb{R}^n$.

1. Introduction

We consider the following equation

$$\begin{cases} u_{tt}(t,x) = Au(t,x) & t \in \mathbb{R},\ x \in \mathbb{R}^n \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $A = \sum_{j,k=1}^n \frac{\partial}{\partial x_j} a_{jk}(x) \frac{\partial}{\partial x_k}$. We assume that there are $c_0 > 0$ and $R > 0$ such that the coefficients $a_{jk}(x) \in C^\infty(\mathbb{R}^n)$ are real valued, satisfy $a_{jk}(x) = a_{kj}(x)$ and

$$a(x,\xi) = \sum_{j,k=1}^n a_{jk}(x)\xi_j\xi_k \geq c_0|\xi|^2, \tag{1.2}$$

for $x, \xi \in \mathbb{R}^n$, and

$$a_{jk}(x) = \delta_{jk}, \tag{1.3}$$

for $|x| \geq R$. Moreover we assume that there is a real-valued function $q \in C^\infty(\mathbb{R}^{2n})$ such that with $C_{\alpha\beta} > 0, C_1 > 0, C_2 \geq 0$

$$|\partial_\xi^\alpha \partial_x^\beta q(x,\xi)| \leq C_{\alpha\beta}(1+|\xi|)^{-|\alpha|}(1+|x|)^{1-|\beta|},\ x,\xi \in \mathbb{R}^n, \tag{1.4}$$

for all α, β, and

$$H_a q = \sum_{j=1}^n \{\partial_\xi a(x,\xi)\partial_x q(x,\xi) - \partial_x a(x,\xi)\partial_\xi q(x,\xi)\} \geq C_1|\xi| - C_2,\ x,\xi \in \mathbb{R}^n. \tag{1.5}$$

This condition is equivalent to the non-trapping condition. See [1] and [4].

Let $\mu \in \mathbb{R}$ and $1 \leq p \leq \infty$ and let L^p be the set of measurable functions over $\mathbb{R}^n$ with integrable pth power. We denote by $W^{l,p}_\mu$ the set of functions $u(x)$

defined in $\mathbb{R}^n$ such that $(1+|x|)^\mu \partial_x^\alpha u(x)$ is contained in L^p for $|\alpha| \le l$. For brevity we denote $L^p_\mu = W^{0,p}_\mu, W^{l,p} = W^{l,p}_0$, $H^l = W^{l,2}$.

Our main theorem is the following.

Theorem 1.1. *Assume that $n \ge 3$ and (1.2)–(1.5) are valid. Let l be a positive integer, $\mu > 1/2$ and $\varepsilon \in (0,1)$. Then there is $C_{l\varepsilon} > 0$ such that for any initial data $u_i (i = 0,1)$ which belong to $H^{l+2n+6} \cap W^{l+2n+6,1}_{1-\varepsilon}$ there is a solution $u(t)$ of (1.1) satisfying*

$$||u_t(t)||_{W^{l,\infty}_{-n/2-3+\varepsilon}} + ||\nabla_x u(t)||_{W^{l,\infty}_{-n/2-3+\varepsilon}} \le C_{l\varepsilon}(1+|t|)^{-\frac{n+1}{2}+\varepsilon}\{||u_0||_{W^{l+2n+6,1}_{1-\varepsilon}} + ||u_1||_{W^{l+2n+6,1}_{1-\varepsilon}}\}, \tag{1.6}$$

for $t \in \mathbb{R}$.

Applying the cut-off method introduced by Shibata and Tsutsumi in [8], we can obtain the uniform decay estimates of solutions of (1.1).

Theorem 1.2. *Assume that $n \ge 3$ and (1.2)–(1.5) are valid and let l be a positive integer. Then there is $C_l > 0$ such that for any initial data $u_i (i = 0,1)$ which belong to $H^{l+n+2} \cap W^{l+2n+6,1}$ there is a solution $u(t)$ of (1.1) which satisfies*

$$||u_t(t)||_{W^{l,\infty}} + ||\nabla_x u(t)||_{W^{l,\infty}} \le C_l(1+|t|)^{-\frac{n-1}{2}}(||u_0||_{W^{l+2n+6,1}} + ||u_1||_{W^{l+2n+6,1}}), \tag{1.7}$$

for $t \in \mathbb{R}$.

2. Wave operators

Let Δ be the Laplacian in $\mathbb{R}^n$, a self-adjoint operator with the definition domain $D(\Delta) = H^2$. We introduce the wave operators between A and Δ defined by

$$W_\pm = \lim_{t\to\pm\infty} e^{-itA}e^{it\Delta}, \tag{2.1}$$

which is a unitary operator in L^2. Denote $H = \sqrt{-A}$ and $H_0 = \sqrt{-\Delta}$. Then the solution $u(t,x)$ of (1.1) can be represented by

$$(Hu(t,x), u_t(t,x)) = (e^{itH}P^{(1)} + e^{-itH}P^{(2)})(Hu_0(x), u_1(x)), \tag{2.2}$$

where $P^{(1)}, P^{(2)}$ are 2×2 matrices,

$$P^{(1)} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix} \tag{2.3}$$

and $P^{(2)} = I - P^{(1)}$, where I is the unit matrix. Using the intertwining and unitary properties of the wave operators $W_\pm$ defined by (2.1), we get

$$e^{itH}\varphi = W_\pm e^{itH_0}W_\pm^*\varphi, \tag{2.4}$$

where $W_\pm^*$ stands for the adjoint operator of $W_\pm$. Therefore we can write

$$(Hu(t,x), u_t(t,x)) = W_\pm(e^{itH_0}P^{(1)} + e^{-itH_0}P^{(2)})W_\pm^*(Hu_0(x), u_1(x)) \tag{2.5}$$

We can prove

Theorem 2.1. *For any integers $l, \mu > 1/2$ and for any $\varepsilon \in (0,1)$ there is $C_{l\mu\varepsilon} > 0$ such that*

$$||W_{\pm}\varphi||_{W^{l,\infty}_{-\mu-n/2-3+\varepsilon}} \le C_{l\varepsilon}||\varphi||_{W^{l+n+3,\infty}_{\varepsilon-1}} \tag{2.6}$$

for $\varphi \in W^{l+n+3,\infty}_{\varepsilon}$ and

$$||W^*_{\pm}\psi||_{W^{l,1}_{1-\varepsilon}} \le C_{l\varepsilon}||\psi||_{W^{l+n+3,1}_{\mu+n/2+3-\varepsilon}} \tag{2.7}$$

for $\psi \in W^{l+n+3,1}_{1-\varepsilon}$.

The proof of this theorem will be given in Section 2.

Proposition 2.2. *Let ε be in $[-\frac{n+1}{2}, \frac{n-1}{2}]$. Then for any nonnegative integer l there is $C_l > 0$ such that*

$$||e^{itH_0}f||_{W^{l,\infty}_{\varepsilon}} \le C_l(1+|t|)^{-\frac{n-1}{2}+\varepsilon}||f||_{W^{l+n+1,1}_{|\varepsilon|}}, \tag{2.8}$$

for $f \in W^{l+n+1,1}_{\varepsilon}$.

Proof. We have

$$e^{itH_0}f(x) = \int_{\mathbb{R}^n} K(x-y,t)\langle D_y\rangle^{n+1}f(y)dy, \tag{2.9}$$

where

$$K(z,t) = \int e^{iz\xi+it|\xi|}\langle\xi\rangle^{-n-1}d\xi \tag{2.10}$$

and we use the notation $\langle D\rangle = (1-\Delta)^{\frac{1}{2}}$ and $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$. Let $|z| \le \frac{|t|}{2}$. Denote $w = \frac{\xi}{|\xi|}$. Then $|zw+t| \ge \frac{|t|}{2}$. Hence

$$K(z,t) = \int_0^\infty \int_{|w|=1} e^{i(zw+t)\rho}\rho^{n-1}\langle\rho\rangle^{-n-1}dwd\rho \tag{2.11}$$

$$= \int_{|w|=1}\left\{\int_0^\infty (zw+t)^{-n}e^{i(zw+t)\rho}(i\partial_\rho)^n(\rho^{n-1}\langle\rho\rangle^{-n-1})d\rho + (n-1)!(-i)^{n-1}\right\}dw$$

and consequently we get

$$|K(z,t)| \le C|t|^{-n} \tag{2.12}$$

for $|z| \le \frac{|t|}{2}$. For $|z| \ge \frac{|t|}{2}$ we can see easily that

$$\left|\int_{0\le\rho\le|z|^{-n}}\int_{|w|=1} e^{i(zw+t)\rho}\rho^{n-1}\langle\rho\rangle^{-n-1}dwd\rho\right| \le C|z|^{-n} \le C|t|^{-n}. \tag{2.13}$$

On the other hand, when $|z|\rho \ge 1$, by use of the stationary phase method we can estimate

$$\left|\int_{|w|=1} e^{izw\rho}dw\right| \le C(|z|\rho)^{-\frac{n-1}{2}}. \tag{2.14}$$

Hence we get

$$\left|\int_{\delta}^{\infty}\int_{|w|=1} e^{i(zw+t)\rho}\rho^{n-1}\langle\rho\rangle^{-n-1}dwd\rho\right| \le C|z|^{-\frac{n-1}{2}} \le C|t|^{-\frac{n-1}{2}+\varepsilon}|z|^{-\varepsilon}, \quad (2.15)$$

for $|z|\rho \ge 1$ (denote $\delta = |z|^{-1}$) and for $\varepsilon \le \frac{n-1}{2}$. Taking account of $(1+|z|)^{\pm 1} \le 2(1+|x|)^{\pm 1}(1+|y|)$ we obtain (2.8) from (2.9), (2.12), (2.13) and (2.15). □

Now we can prove Theorem 1.1 by use of Theorem 2.1. In fact, from (2.5) we easily obtain (1.7) by use of Theorem 2.1 and Proposition 2.2.

3. Integral representation of wave operators

First we mention a well known result which can be found for example in the textbook of Mochizuki [5].

Proposition 3.1. *The wave operators $W_\pm$ have the following integral representation*

$$W_\pm\varphi(x) = (2\pi)^{-n}\int e^{ix\xi}w_\pm(x,\xi)\hat{\varphi}(\xi)d\xi, \quad (3.1)$$

where

$$w_\pm(x,\xi) = 1 - w_\pm^1(x,\xi) \quad (3.2)$$

and

$$w_\pm^1(x,\xi) = (A + (|\xi| \pm i0)^2)^{-1}\sum_{l,k=1}^{n}\{(a_{jk}(x)-\delta_{jk})\xi_l\xi_k + \partial_{x_l}a_{lk}(x)\xi_k\}. \quad (3.3)$$

Besides we can investigate more precisely the properties of the symbol $w_\pm^1(x,\xi)$.

Theorem 3.2. *Assume that $n \ge 3$ and* (1.2)–(1.5) *are valid. Then the symbol $w_\pm^1$ satisfies*

$$|\partial_\xi^\alpha\partial_x^\beta w_\pm^1(x,\xi)| \le C_{\alpha\beta\mu}|\xi|^{1-|\alpha|}\langle\xi\rangle\langle x\rangle^{\mu+1/2+|\alpha|}, \quad (3.4)$$

for $\alpha,\beta \in Z_+^n, \mu > 1/2$.

The proof of this theorem will be given in Section 3.

Now we can prove Theorem 2.1.

Proof of Theorem 2.1. Let χ be a cut-off function such that $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = 0$ for $t \ge 2$ and $l_0 = n+3$. Let $W_\pm^1$ be the pseudo-differential operator with symbol $w_\pm^1$. Then

$$\begin{aligned} W_\pm^1 f(x) &= \int e^{i(x-y)\xi}w_\pm^1(x,\xi)\langle\xi\rangle^{-l_0}\chi(|\xi|)d\xi\langle D\rangle^{l_0}f(y)dy \quad (3.5)\\ &\quad + \int e^{i(x-y)\xi}w_\pm^1(x,\xi)\langle\xi\rangle^{-l_0}(1-\chi(|\xi|))d\xi\langle D\rangle^{l_0}f(y)dy \\ &=: \int K_0(x-y)\langle D\rangle^{l_0}f(y)dy + \int K_1(x-y)\langle D\rangle^{l_0}f(y)dy. \end{aligned}$$

Then

$$K_1(z) = \langle z\rangle^{-l} \int e^{iz\xi} \langle D_\xi\rangle^l \left\{ w^1_\pm(x,\xi)\langle\xi\rangle^{-l_0}(1-\chi(|\xi|)) \right\} d\xi, \tag{3.6}$$

which satisfies from (3.4) together with $|\xi| \geq 1$

$$|K_1(z)| \leq C\langle z\rangle^{-l}\langle x\rangle^{\mu+1/2+l}, \tag{3.7}$$

for any positive even integer l. On the other hand

$$\begin{aligned} K_0(z) &= \int e^{iz\xi} w^1_\pm(x,\xi)\langle\xi\rangle^{-l_0}\chi(|\xi|)d\xi \qquad (3.8)\\ &= \int_{|\xi|\leq 2|z|^{-1}} e^{iz\xi} w^1_\pm(x,\xi)\langle\xi\rangle^{-l_0}\chi(|\xi|)\chi(|z||\xi|)d\xi \\ &\quad + \int_{|\xi|\geq |z|^{-1}} e^{iz\xi} w^1_\pm(x,\xi)\langle\xi\rangle^{-l_0}\chi(|\xi|)(1-\chi(|z||\xi|))d\xi \\ &=: K_{00}(z) + K_{01}(z). \end{aligned}$$

We can see easily from (3.4) that

$$|K_{00}(z)| \leq C|z|^{-n-1}\langle x\rangle^{\mu+1/2} \leq C|z|^{-n-\varepsilon}\frac{\langle x\rangle^{\mu+1/2+1-\varepsilon}}{|y|^{1-\varepsilon}}. \tag{3.9}$$

Using a polar coordinate, we have

$$\begin{aligned} K_{01}(z) = \int_{\rho\geq|z|^{-1}} \int_{|w|=1} & e^{izw\rho} w^1_\pm(x,\rho w)\rho^{n-1} \qquad (3.10)\\ & \times\langle\rho\rangle^{-l_0}\chi(\rho)(1-\chi(|z|\rho))dwd\rho. \end{aligned}$$

By use of the stationary phase method (for example, see [2]) we can derive

$$\int_{|w|=1} e^{izw\rho} w^1_\pm(x,\xi)dw = (|z|\rho|)^{-\frac{n-1}{2}} \left\{ e^{i|z|\rho} q^1_\pm(x,\rho) + e^{-i|z|\rho} q^2_\pm(x,\rho) \right\}$$

where $q^j_\pm(x,\rho)$ satisfies from (3.4)

$$|\partial^k_\rho q^j_\pm(x,\rho)| \leq C_k\rho^{1-k}|x|^{\mu+1/2+k}, \tag{3.11}$$

for $|z|\rho \geq 1, j = 1,2$ and $k = 0,1,\ldots$. Hence we can write

$$\begin{aligned} K_{01}(z) &= |z|^{-\frac{n-1}{2}} \int_{\rho\geq|z|^{-1}} e^{i|z|\rho} q^1_\pm(x,\rho)\rho^{\frac{n-1}{2}} \\ &\qquad \times\langle\rho\rangle^{-l_0}\chi(\rho)(1-\chi(|z|\rho))d\rho \\ &\quad +|z|^{-\frac{n-1}{2}} \int_{\rho\geq|z|^{-1}} e^{-i|z|\rho} q^2_\pm(x,\rho)\rho^{\frac{n-1}{2}} \qquad (3.12)\\ &\qquad \times\langle\rho\rangle^{-l_0}\chi(\rho)(1-\chi(\rho|z|))d\rho \\ &=: K^1(z) + K^2(z). \end{aligned}$$

So we can see that

$$K^1(z) = |z|^{-\frac{n-1}{2}-l} \int e^{i|z|\rho}(i\partial_\rho)^l\{\rho^{\frac{n-1}{2}} q^1_\pm(x,\rho)\langle\rho\rangle^{-l_0}\chi(\rho)(1-\chi(\rho|z|))\}d\rho. \quad (3.13)$$

We take $l = \frac{n-1}{2} + 2$, if n is odd and $l = \frac{n+2}{2} + 1$, if n is even. Then we get from (3.13) by virtue of (3.11)

$$|K^1(z)| \le C|z|^{-n-1}\log|z||x|^{\mu+n/2+2} \le C|z|^{-n-\varepsilon}\frac{|x|^{\mu+n/2+3-\varepsilon}}{|y|^{1-\varepsilon}}, \quad (3.14)$$

for any $\varepsilon > 0$. Analogously $K^2(z)$ satisfies (3.14). Taking account of $|z| = |x-y| \le \langle x\rangle\langle y\rangle$, we can see that (3.7), (3.9) and (3.14) for $K^i(i=1,2)$ imply (2.6). □

4. Estimate of resolvents

In this section we shall prove Theorem 3.2. Let $\lambda = \sigma \pm i\varepsilon, \varepsilon > 0, R_0(\lambda^2) = (\Delta + \lambda^2)^{-1}$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then

$$R_0(\lambda^2)f(x) = \int K(x-y,\lambda)f(y)dy, \quad (4.1)$$

where

$$K(z,\lambda) = \int e^{iz\xi}(\lambda^2 - |\xi|^2)^{-1}d\xi. \quad (4.2)$$

We can calculate $K(z,\lambda)$ as follows. We denote $r = |z|$. Then, for $\Im\lambda > 0$

$$K(z,\lambda) = \frac{i}{2\lambda}e^{i\lambda r}, \quad (n=1) \quad (4.3)$$

$$K(z,\lambda) = \left(\frac{-1}{2\pi r}\frac{d}{dr}\right)^{p-1}\left[\frac{i}{4}H_0^{(1)}(\lambda r)\right], \quad (n=2p), \quad (4.4)$$

and

$$K(z,\lambda) = \left(\frac{-1}{2\pi r}\frac{d}{dr}\right)^{p-1}\left[\frac{e^{i\lambda r}}{r}\right], \quad (n=2p+1), \quad (4.5)$$

here $H_0^{(0)}(\zeta)$ ia a Henkel's function and $p = 1, 2, \ldots$. Noting that

$$\left(\frac{1}{r}\partial_r\right)^{p-1} = \sum_{l=1}^{p-1} c_l^{p-1} r^{-2k+l}\partial_r^l, \quad (4.6)$$

in the case of the odd dimension we can see easily from (4.5) that K satisfies

$$K(z,\lambda) = \sum_{l=1}^{p-1} c_l^{p-1} r^{-2k+l}\partial_r^l\left[\frac{e^{i\lambda r}}{r}\right]. \quad (4.7)$$

Then we can see easily that for $n \ge 3$

$$|\partial_\lambda^k K(z,\lambda)| \le C_k r^{-\frac{n-1}{2}+k}, \quad (4.8)$$

and

$$|\partial_\lambda^k(\partial_r \mp i\lambda)K(z,\lambda)| \le C_k r^{-\frac{n-1}{2}-1+k}, \tag{4.9}$$

for $|\lambda| \le 1$, where we denote $r = |z|$. In fact, in the case of the even dimension we have for $p \ge 2$ from (4.7)

$$K(z,\lambda) = \sum_{l=1}^{p-1} c_l^{p-1} r^{-2(p-1)+l} \lambda^l [\partial_\zeta^l H_0^{(1)}](r\lambda). \tag{4.10}$$

The expression of $H_0^{(1)}(\zeta)$ is given by

$$H_0^{(1)}(\zeta) = \int_0^\infty \frac{\rho}{\zeta^2 - \rho^2} \int_0^\pi e^{i\rho\cos\theta} d\theta d\rho. \tag{4.11}$$

From this expression we can see easily that

$$|\zeta^l \partial_\zeta^l H_0^{(1)}(\zeta)| \le C_l(1+|\zeta|)^l, \tag{4.12}$$

for $\zeta \in \mathbb{C}$ with $\Im\zeta \ge 0$ and $l = 1, 2, \ldots$, where C_l is independent of ζ. From the representations (4.10) and (4.12) we can show that (4.8) holds in the case of the even dimension. The estimate (4.9) is well known for $|\lambda| r \ge 1$ and is derived from (4.12) immediately. Thus we get

Proposition 4.1. *Let $\lambda = \sigma \pm i\varepsilon, \varepsilon > 0$, $|\lambda| \le 1$ and $n \ge 3$. Then*

$$\left|\partial_\lambda^k K(z,\lambda)\right| \le C_k r^{-\frac{n-1}{2}+k}, \tag{4.13}$$

and

$$\left|\partial_\lambda^k(\partial_r \mp i\lambda)K(z,\lambda)\right| \le C_k r^{-\frac{n-1}{2}-1+k}, \tag{4.14}$$

for $k = 0, 1, \ldots,$ where C_k is independent of $|\lambda| \le 1$ and $r = |z|$.

Now we can prove

Proposition 4.2. *Let $\lambda = \sigma \pm i\varepsilon, \varepsilon > 0$, $|\lambda| \le 1, \mu > \frac{1}{2}$ and $f \in C_0^\infty(\mathbb{R}^n)$. Then there is $R_1 > 0$ such that for $|x| \ge R_1$ and for $k = 0, 1, \ldots,$*

$$|R_0(\lambda^2)^{k+1} f(x)| \le \frac{C_k |x|^k}{|x|^{\frac{n-1}{2}} |\lambda|^k} ||f||_{L^2}, \tag{4.15}$$

$$|(\partial_r \mp i\lambda) R_0(\lambda^2)^{k+1} f(x)| \le \frac{C_k |x|^k}{|x|^{\frac{n-1}{2}+1} |\lambda|^k} ||f||_{L^2}, \tag{4.16}$$

$$||R_0(\lambda^2)^{k+1} f||_{L^2_{-\mu-k}} \le \frac{C_k}{|\lambda|^k} ||f||_{L^2}, \tag{4.17}$$

and

$$||(\partial_r \mp i\lambda) R_0(\lambda^2)^{k+1} f||_{L^2_{-\mu-k+1}} \le \frac{C_k}{|\lambda|^k} ||f||_{L^2}, \tag{4.18}$$

where $C_k > 0$ is independent of $|\lambda| \le 1$ but depends on the support of f.

Proof. Differentiating the relation (4.1) with respect to λ we get

$$R_0(\lambda^2)^{k+1} f(x) = \int \left(\frac{1}{\lambda}\partial_\lambda\right)^k K(x-y,\lambda) f(y)dy. \tag{4.19}$$

Noting that

$$\left(\frac{1}{\lambda}\partial_\lambda\right)^k = \sum_{l=1}^{k} c_l^k \lambda^{-2k+l} \partial_\lambda^l, \tag{4.20}$$

we get

$$R_0(\lambda^2)^{k+1} f(x) = \int \sum_{l=1}^{k} c_l^k \lambda^{-2k+l} \partial_\lambda^l K(x-y,\lambda) f(y)dy. \tag{4.21}$$

Hence by virtue of (4.13) and (4.21) we can obtain (4.15). We get also (4.17) integrating (4.15). Analogously we get (4.16) and (4.18) from (4.14). $\square$

Next we consider the equation

$$(A+\lambda^2)u(\lambda,x) = f(x), \quad x \in \mathbb{R}^n, \tag{4.22}$$

Let $\lambda = \sigma \pm i\varepsilon, \varepsilon > 0$, $|\lambda| \leq 1$ and $f \in C_0^\infty(\Omega)$. Then we can get

$$u(\lambda,x) = (A+\lambda^2)^{-1} f(x), \quad x \in \mathbb{R}^n. \tag{4.23}$$

Lemma 4.3. *Let $\lambda = \sigma \pm i\varepsilon, \varepsilon > 0$, $|\lambda| \leq 1$ and $f \in C_0^\infty(\mathbb{R}^n), \beta(x) \in C^\infty(\mathbb{R}^n)$ such that $\beta(x) = 1$ for $|x| \geq R+1, \beta(x) = 0$ for $|x| \leq R$, where we take $R > 0$ given in (1.3) and denote $u^k(\lambda,x) = \partial_\lambda^k u(\lambda,x)$. Then*

$$\beta(x)u^k(\lambda,x) = \sum_{l=0}^{k} P_l^k V u^l + Q^k \beta f, \tag{4.24}$$

where $V = -[A,\beta], P_0^0 = R_0, Q^0 = R_0, P_1^1 = R_0, P_0^1 = -2\lambda R_0^2, Q^1 = -2\lambda R_0^2$. For $k \geq 2$ if l is odd

$$P_{k-l}^k = \lambda R_0(\lambda^2)^{\frac{l+3}{2}} \sum_{j=0}^{\frac{l-1}{2}} c_{lj}^k \lambda^{2j} R_0^j, \tag{4.25}$$

and if l is even

$$P_{k-l}^k = R_0(\lambda^2)^{\frac{l+2}{2}} \sum_{j=0}^{\frac{l}{2}} c_{lj}^k \lambda^{2j} R_0^j, \tag{4.26}$$

where c_{lj}^k are constants and $R_0 = (\Delta+\lambda^2)^{-1}$. If $k \geq 3$ is odd

$$Q^k = \lambda R_0^{\frac{n+1}{2}} \sum_{j=0}^{\frac{k-1}{2}} d_j^k \lambda^{2j} R_0^j \tag{4.27}$$

and if $k \geq 2$ is even

$$Q^k = R_0^{\frac{n}{2}+1} \sum_{j=0}^{\frac{k}{2}} d_j^k \lambda^{2j} R_0^j, \tag{4.28}$$

where d_j^k are constants.

Proof. In the case $k = 0$ (4.24) is trivial. Differentiating the equation (4.22) we get

$$(A + \lambda^2)u^1(\lambda, x) = -2\lambda u. \tag{4.29}$$

Hence (1.3) yields

$$(\Delta + \lambda^2)\beta u^1 = Vu^1 - 2\lambda\beta u = Vu^1 - 2\lambda R_0(Vu + \beta f), \tag{4.30}$$

which proves that $P_1^1 = R_0, P_0^1 = -2\lambda R_0^2, Q^1 = R_0^2$, that is, the case $k = 1$. For $k \geq 2$ u^k satisfies

$$(A + \lambda^2)u^k = -k\lambda u^{k-1} - k(k-1)u^{k-2}. \tag{4.31}$$

Hence we can prove (4.24) by induction on k. $\square$

We need a priori estimates for Δ in weighted Sobolev spaces.

Lemma 4.4. *Let $\mu \in \mathbb{R}$ and u be in L^2_μ and Au in L^2_μ. Then u is in $W^{2,2}_\mu$ and satisfies*

$$||u||_{W^{2,2}_\mu} \leq C \left(||Au||_{L^2_\mu} + ||u||_{L^2_\mu} \right). \tag{4.32}$$

Now we can prove

Proposition 4.5. *Let $\lambda = \sigma \pm i\varepsilon, \varepsilon > 0$, $|\lambda| \leq 1$ and $f \in C_0^\infty(\mathbb{R}^n), \mu > \frac{1}{2}, 0 < \delta < \mu - \frac{1}{2}$ and u be a solution of the equation (4.22). Denote $u^k = \partial_\lambda^k u$. Then there is $R_0 > 0$ (given in Proposition 4.2) such that*

$$|u^k(\lambda, x)| \leq C_k \frac{|x|^{\frac{-n+1+k}{2}}}{|\lambda|^{\frac{k}{2}}} \left(\sum_{j=0}^{k} \left(\frac{|\lambda|}{|x|} \right)^{\frac{j}{2}} ||u^j||_{L^2_{-\mu-\frac{j}{2}}} + ||f||_{L^2} \right), \tag{4.33}$$

$$|(\partial_r \pm i\lambda)u^k(\lambda, x)| \leq C_k \frac{|x|^{\frac{-n+1+k}{2}-1}}{|\lambda|^{\frac{k}{2}}} \left(\sum_{j=0}^{k} \left(\frac{|\lambda|}{|x|} \right)^{\frac{j}{2}} ||u^j||_{L^2_{-\mu-\frac{j}{2}}} + ||f||_{L^2} \right), \tag{4.34}$$

for $|x| \geq R_0$ and $k = 0, 1, \ldots,$ where $C_k > 0$ is independent of λ but depends on the support of f, and for any $R_1 \geq R_0$

$$||u^k(\lambda)||_{W^{l,2}_{-\mu-\frac{k}{2}}(B_{R_1})} \leq \frac{C_{kl}}{R_1^\delta |\lambda|^{\frac{k}{2}}} \left(\sum_{j=0}^{k} |\lambda|^{\frac{j}{2}} ||u^j||_{W^{l,2}_{-\mu-\frac{j}{2}}} + ||f||_{W^{l,2}} \right), \tag{4.35}$$

$$||(\partial_r \pm i\lambda)u^k(\lambda)||_{W^{l,2}_{-\mu-\frac{k}{2}+1}(B_{R_1})} \leq \frac{C_{kl}}{R_1^\delta |\lambda|^{\frac{k}{2}}} \left(\sum_{j=0}^{k} |\lambda|^{\frac{j}{2}} ||u^j||_{W^{l,2}_{-\mu-\frac{j}{2}}} + ||f||_{W^{l,2}} \right), \tag{4.36}$$

for $k = 0, 1, \ldots$ and $l = 0, 1, \ldots,$ where $B_{R_1} = \{x \in \mathbb{R}^n; |x| \geq R_1\}$ and $C_{kl} > 0$ is independent of λ and R_1 but depends on the support of f.

Proof. It follows from (4.24) that $\beta(x) = 1$ for $|x| \geq R$ implies

$$u^k(\lambda, x) = \sum_{l=0}^{k} P^k_{k-l} V u^{k-l} + Q^k \beta f, \tag{4.37}$$

where $V = -[A, \beta]$ is a first order differential operator whose coefficients have a compact support. By use of (4.25), (4.26) and (4.15) we get

$$|P^k_{k-l} V u^{k-l}(\lambda, x)| \leq C_k \frac{|x|^{-\frac{n-1}{2}+\frac{l}{2}}}{|\lambda|^{\frac{l}{2}}} ||V u^{k-l}||_{L^2}. \tag{4.38}$$

Besides, it follows from (4.31) and Lemma 4.4 that

$$\begin{aligned} ||V u^{k-l}||_{L^2(\mathbb{R}^n)} &\leq C ||u^{k-l}||_{W^{1,2}_{-\mu-\frac{l}{2}}} \\ &\leq C(||u^{k-l}||_{L^2_{-\mu-\frac{l}{2}}} \\ &\quad + ||(k-l)\lambda u^{k-l} + (k-l)(k-l-1)u^{k-l-2}||_{L^2_{-\mu-\frac{l}{2}}}. \end{aligned} \tag{4.39}$$

Hence we get (4.33) from (4.37) and (4.38) by use of (4.17). Analogously we get (4.34) by use of (4.18). The integration over B_R of (4.33) and (4.34) implies (4.35) and (4.36) for $l = 0$ respectively. For $l \geq 1$ we put $u^k = \partial^k_\lambda (\sqrt{-A})^l u(\lambda)$ and reduce to the case $l = 0$, Then we can show (4.35) and (4.36) for $l \geq 1$ analogously to the case $l = 0$. □

Now we can prove

Theorem 4.6. *Let $\lambda = \sigma \pm i\varepsilon, \varepsilon \geq 0$, $0 < |\lambda| \leq 1, l \geq [\frac{n}{2}] + 1$ and $f \in C_0^\infty(\mathbb{R}^n)$ with* supp$f \subset K$ *(K: a compact set in $\mathbb{R}^n$), $\mu > \frac{1}{2}$ and u be a solution of the equation (4.22). Denote $u^k = \partial^k_\lambda u$. Then there is $C_{kl}(K) > 0$ which is independent of λ such that*

$$||u^k(\lambda)||_{W^{l,2}_{-\mu-\frac{k}{2}}} \leq \frac{C_{kl}(K)}{|\lambda|^k} ||f||_{H^l}, \tag{4.40}$$

for $k, l = 0, 1, \ldots$

Proof. First of all we prove that A has neither zero eigenvalue nor zero resonance. In fact, assume $v(x)$ satisfies $Av = 0$ and

$$|v(x)| \leq C|x|^{\frac{-n+1}{2}}, \tag{4.41}$$

$$|\partial_r v(x)| \leq C|x|^{\frac{-n+1}{2}-1}, \tag{4.42}$$

for $|x| \geq R_0$. Then Green's formula gives

$$
\begin{aligned}
0 &= \int_{|x|\leq r} Av(x)\overline{v(x)}dx \qquad (4.43)\\
&= \int_{|x|\leq r} \sum a_{jl}(x)D_jv(x)\overline{D_lv(x)}dx \\
&\quad + \int_{|x|=r} \sum a_{jl}(x)D_jv(x)\overline{D_lv(x)}dx \\
&\geq c_0 \int_{|x|\leq r} |\nabla_x v(x)|^2 dx + \int_{|x|=r} \sum a_{jl}(x)\frac{x_j}{|x|}v(x)\overline{D_lv(x)}dS_r.
\end{aligned}
$$

Besides taking account of (4.41) and (4.42) we can see that

$$\lim_{r\to\infty} \int_{|x|=r} \sum a_{jl}(x)\frac{x_j}{|x|}v(x)\overline{D_lv(x)}dS_r = 0,$$

which implies together with (4.43) that $\nabla_x v(x)$ vanishes identically in $\mathbb{R}^n$ and consequently $v(x) = 0$ follows from (4.41). Therefore (4.40) for $k = 0$ follows from the standard method (Eidus' method).

For $k \geq 1$ we shall prove (4.40) by induction. Assume that $u^j(j = 0, 1, \ldots, k-1)$ satisfy (4.40) but u^k does not satisfy, that is, there are sequences $\{u^k(\lambda_j)\}_{j=1}^{\infty}$ and $\{f_j\}_{j=1}^{\infty}$ such that $||\lambda_j^k u^k(\lambda_j)||_{W^{l_0,2}_{-\mu-\frac{k}{2}}} = 1$ and $\mathrm{supp} f_j \subset K$, $||f_j||_{H^l} \to 0$ and $\lambda_j \to 0$ $(j \to \infty)$. Put $v_j^k = \lambda_j^k u^k(\lambda_j)$. Then it follows from (4.35) and Reillich's Theorem that $\{v_j^k\}$ has a convergent subsequence in $W^{l,2}_{-\mu-\frac{k}{2}}$ whose limit is denoted by $v^k(0,x)$. For simplicity we denote by the same notation $\{v_j^k\}$ a convergent subsequence which satisfies from (4.31)

$$(A + \lambda_j^2)v_j^k = \lambda_j^2(kv_j^{k-1} + k(k-1)v_j^{k-2}) =: g_j^k. \qquad (4.44)$$

The inductive assumption and $||f_j||_{H^l} \to 0$ $(j \to \infty)$ imply $||g_j^k||_{W^{l,2}_{-\mu-\frac{k}{2}}} \to 0$ $(j \to \infty)$. Therefore (4.44) yields

$$Av^k(0,x) = 0. \qquad (4.45)$$

On the other hand from the Sobolev lemma, v_j^k satisfies for any $\varepsilon > 0$

$$
\begin{aligned}
\langle x\rangle^{-\mu-\frac{k}{2}}|v^k(0,x)| &\leq \langle x\rangle^{-\mu-\frac{k}{2}}(|v^k(0,x) - v_j^k(x)| + |v_j^k(x)|) \qquad (4.46)\\
&\leq C||v^k(0,\cdot) - v_j^k(\cdot)||_{W^{[\frac{n}{2}]+1,2}_{-\mu-\frac{k}{2}}} + \langle x\rangle^{-\mu-\frac{k}{2}}|v_j^k(x)| \\
&\leq \varepsilon + \langle x\rangle^{-\mu-\frac{k}{2}}|v_j^k(x)|.
\end{aligned}
$$

Besides $v_j^k(x)$ satisfies from (4.33)

$$|v_j^k(x)| = |\lambda_j^k u(\lambda_j, x)| \leq C|x|^{-\frac{n-1-k}{2}}|\lambda_j|^{\frac{k}{2}} \leq C|x|^{-\frac{n-1}{2}}, \qquad (4.47)$$

for $R_0 \le |x| \le |\lambda_j|^{-1}$. Hence we have from (4.46), (4.47)

$$|v^k(0,x)| \le C|x|^{-\frac{n-1}{2}} + \varepsilon\langle x\rangle^{\mu+\frac{k}{2}}, \tag{4.48}$$

for $R_0 \le |x| \le |\lambda_j|^{-1}$. Since $\varepsilon > 0$ is arbitrary and $\lambda_j \to 0$ $(j \to \infty)$ we get

$$|v^k(0,x)| \le C|x|^{-\frac{n-1}{2}}, \tag{4.49}$$

for any $|x| \ge R_0$. Similarly we get from (4.34)

$$|\partial_r v^k(0,x)| \le C|x|^{-\frac{n-1}{2}-1}, \tag{4.50}$$

for any $|x| \ge R_0$. Therefore $v^k(0,x)$ is a zero resonance of A, that is, $v^k(0,x)$ vanishes identically in $\mathbb{R}^n$. This is a contradiction to $||v^k(0)||_{W^{l,2}_{-\mu-\frac{k}{2}}} = 1$. Thus we have proved (4.40). □

Remark 4.7. Theorem 4.6 is a special case of Theorem 4.3 by Murata [6] who treated the case that the operator A has zero eigenvalue or zero resonance.

For $|\lambda| \ge 1$ we can derive the estimate of the resolvent as follows.

Theorem 4.8. *Assume that there is a real-valued function $q \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ satisfying* (1.4) *and* (1.5). *Let $\lambda = \sigma \pm i\varepsilon, \varepsilon \ge 0, |\lambda| \ge 1,\ f \in C_0^\infty(\mathbb{R}^n), \mu > \frac{1}{2}$ and u be a solution of the equation* (4.22). *Then there are $C_l > 0$ and $\hat{l}(= C\mu) > 0$ which is independent of λ such that*

$$||u(\lambda)||_{W^{l,2}_{-\mu-1/2}} \le C_l|\lambda|^{-1}||f||_{W^{l+1,2}_{-\mu+1/2+\hat{l}}}, \tag{4.51}$$

for $|\lambda| \ge 1$.

Proof. It is well known from the limiting absorption principle (for example, see [5]) that for any $b > a > 0$ there is a positive constant C depending on a, b such that

$$||u(\lambda)||_{W^{l,2}_{-\mu}} \le C||f||_{W^{l,2}_{\mu}}, \tag{4.52}$$

for $b \ge |\lambda| \ge a$ and $\mu > \frac{1}{2}$. Therefore it suffices to prove (4.51) for $|\lambda| \ge \lambda_0$ where $\lambda_0 > 0$ is sufficiently large. Since (4.52) implies $u \in W^{l,2}_{-\mu}$, $v = \langle x\rangle^{-\mu}u$ belongs to $W^{l,2}$ and satisfies the following equation from (4.22)

$$(A + B + \lambda^2)v(\lambda,x) = \langle x\rangle^{-\mu}f(x), \quad x \in \mathbb{R}^n, \tag{4.53}$$

where

$$Bv = i\sum_{j,k=1}^n \left\{\mu\frac{x_j}{\langle x\rangle^2}a_{jk}(x)(D_k + \mu\frac{x_k}{\langle x\rangle^2}) + D_j\{a_{jk}(x)\mu\frac{x_k}{\langle x\rangle^2}v\right\},\ D_j = i^{-1}\partial_j, \tag{4.54}$$

whose symbol $B(x,\xi)$ is a first order polynomial in ξ, $\Re B$ is a function of order -2 in x and the imaginary part of B satisfies

$$|\partial_\xi^\alpha\partial_x^\beta \Im B(x,\xi)| \le C_{\alpha\beta}\mu\langle x\rangle^{-1-|\beta|}\langle\xi\rangle^{1-|\alpha|}, \quad x,\xi \in \mathbb{R}^n \tag{4.55}$$

Here we introduce Hörmander's notation. For a Riemannian metric $g = \langle x\rangle^{-2}dx^2 + \langle\xi\rangle^{-2}d\xi^2$ and a weight function $m = m(x,\xi)$, denote by $S(m,g)$ the set of symbols $a(x,\xi)$ satisfying for any α,β,

$$|\partial_\xi^\alpha \partial_x^\beta a(x,\xi)| \le C_{\alpha\beta} m(x,\xi)\langle x\rangle^{-|\beta|}\langle\xi\rangle^{-|\alpha|}, \quad x,\xi \in \mathbb{R}^n.$$

For example $\Im B(x,\xi) \in S(\langle x\rangle^{-1}\langle\xi\rangle, g)$ and $\Re B \in S(\langle x\rangle^{-2}, g)$.

We want to derive a priori estimates for the equation (4.53) in Sobolev spaces, by use of the function $q(x,\xi)$ satisfying (1.4) and (1.5). Let $\varepsilon > 0$ be a parameter which will be fixed later and take $\phi \in C_0^\infty(\mathbb{R}^n)$ such that $\phi(t) = 0$ if $t \le 1, \phi(t) = 1$ if $t \ge 2$ and $\phi'(x)t \ge 0$ on $\mathbb{R}$. Set $\phi_+(t) = \phi(t/\varepsilon), \phi_-(t) = \phi_+(-t)$ and $\phi_0 = 1 - \phi_+ - \phi_-$. Define $\psi_0(x,\xi), \psi_\pm(x,\xi)$ by $\psi_0 = \phi_0(q(x,\xi)/\langle x\rangle), \psi_\pm = \phi_\pm(q(x,\xi)/\langle x\rangle)$. Put

$$p(x,\xi) = \frac{q\log(1+\langle x\rangle)}{\langle x\rangle}\psi_0 + (\log|q|)(\psi_+ - \psi_-). \tag{4.56}$$

Then we can see from Lemma 2.3 in Doi [1] that p satisfies

$$|\partial_\xi^\alpha \partial_x^\beta p(x,\xi)| \le C_{\alpha\beta}\log(1+\langle x\rangle)\langle\xi\rangle^{-|\alpha|}\langle x\rangle^{-|\beta|}, \quad x,\xi \in \mathbb{R}^n \tag{4.57}$$

for all α,β that is, $p \in S(\log(1+\langle x\rangle, g)$, and if $\varepsilon > 0$ is chosen sufficiently small,

$$H_a p(x,\xi) \ge \frac{c_0(\log(1+\langle x\rangle))(\langle\xi\rangle - C)}{\langle x\rangle} \ge \frac{c_0(\langle\xi\rangle - C)}{\langle x\rangle}, \quad x,\xi \in \mathbb{R}^n. \tag{4.58}$$

Denote for $M \in \mathbb{R}$

$$e^{Mp}(x,D)v(x) = \int_{\mathbb{R}^n} e^{ix\xi}e^{Mp(x,\xi)}\hat{v}(\xi)d\xi, \tag{4.59}$$

which is defined as a pseudo-differential operator because $e^{Mp(x,\xi)} \in S(\langle x\rangle^{C_1 M}, \tilde{g})$, $\tilde{g} = (\log(1+\langle x\rangle))^2 g$ and put

$$w(\lambda,x) = e^{Mp}(x,D)v(\lambda,x), \tag{4.60}$$

which satisfies from (4.53)

$$(A + \tilde{B} + \lambda^2)w(\lambda,x) + R_\infty(x,D)v = e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x), \quad x \in \mathbb{R}^n, \tag{4.61}$$

where $R_\infty(x,\xi) \in \bigcap_{l>0} S((\langle x\rangle\langle\xi\rangle)^{-l}, g)$ and $\tilde{B}$ satisfies

$$\tilde{B} = B + iMH_a p(x,D) + R_0(x,D) \tag{4.62}$$

where $R_0(x,\xi) \in S\left(\left(\frac{\log(1+\langle x\rangle)}{\langle x\rangle}\right)^2, g\right)$ and from (4.58)

$$\begin{aligned} \pm\Im(\tilde{B}(x,\xi) + \lambda^2) &= \pm(\Im B(x,\xi) + MH_a p) + |\sigma|\varepsilon \\ &\ge \{(c_0(\pm M) - C)\langle\xi\rangle - C)\})\langle x\rangle^{-1}, \quad x,\xi \in \mathbb{R}^n, \end{aligned} \tag{4.63}$$

Here we choose $M \in \mathbb{R}$ such that $\pm M > 0$, ($\pm$ meaning the sign of the real part of $\lambda = \sigma$) and $(c_0(\pm M) - C) > 0$. Therefore we can get from (4.63)

$$\pm\Im((A + \tilde{B} + \lambda^2)w, w) \ge \Re(\langle x\rangle^{-1}\{(c_0(\pm M) - C)(\langle D\rangle - C)\}w, w). \tag{4.64}$$

Denote $e_1(\xi) = \chi(\frac{|\xi|}{\varepsilon|\lambda|})$ and $e_0(\xi)^2 = 1 - e_1(\xi)^2$, where $\chi(t) = 1$, if $|t| \le 1$ and $\chi(t) = 0$ if $|t| \ge 2$. Then we have

$$\Re(\langle x\rangle^{-1}\{(c_0M - C)(\langle D\rangle - C\}e_0(D)w, e_0(D)w) \tag{4.65}$$
$$\ge c_1(|\lambda|\langle x\rangle^{-1}e_0(D)w, e_0(D)w),$$

for $|\lambda| \ge \lambda_0$ sufficiently large. On the other hand since $\Re(A+\tilde{B}+\lambda^2)(x,\xi)e_1(\xi) \ge c_2|\lambda|^2 e_1(\xi)$ for $x, \xi \in \mathbb{R}^n$, if ε is small, we get

$$\Re((A+\tilde{B}+\lambda^2)e_1(D)w, e_1(D)w) \ge \frac{c_2}{2}(|\lambda|^2 e_1(D)w, e_1(D)w), \tag{4.66}$$

for $|\lambda| \ge \lambda_0$, if we choose $\lambda_0 > 0$ large. Moreover since from (4.61) we have

$$(A+\tilde{B}+\lambda^2)e_1(D)w(\lambda,x) + e_1(D)R_\infty(x,D)v \tag{4.67}$$
$$= [A+\tilde{B}, e_1]w + e_1(D)e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x), \quad x \in \mathbb{R}^n,$$

we can calculate

$$\Re((A+\tilde{B}+\lambda^2)e_1(D)w, e_1(D)w) \tag{4.68}$$
$$= \Re((-e_1(D)R_\infty(x,D)v + [A+\tilde{B}, e_1]w + e_1(D)e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x)), e_1(D)w)$$
$$\le C(||\langle x\rangle^{-1}|\lambda|w|| + ||e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x)||)||e_1 w||.$$

Hence we get from (4.66) and (4.68)

$$||e_1 w|| \le C(|\lambda|^{-1}||\langle x\rangle^{-1}w|| + |\lambda|^{-2}||e_1(D)e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x)||). \tag{4.69}$$

Besides,

$$\Re(\langle x\rangle^{-1}\{(c_0M - C)(\langle D\rangle - C)\}w, w) \tag{4.70}$$
$$= \Re(\langle x\rangle^{-1}\{(c_0M - C)(\langle D\rangle - C)\}w, (e_0(D)^2 + e_1(D)^2)w)$$
$$\ge \Re(\langle x\rangle^{-1}\{(c_0M - C)(\langle D\rangle - C)\}e_0(D)w, e_0(D)w)$$
$$+\Re(\langle x\rangle^{-1}\{(c_0M - C)(\langle D\rangle - C)\}e_1(D)w, e_1(D)w) - C||\langle x\rangle^{-1/2}w||^2$$
$$\ge c_4|\lambda|(\langle x\rangle^{-1}e_0(D)w, e_0(D)w) - C|\lambda|(\langle x\rangle^{-1}e_1(D)w, e_1(D)w) - C||(\langle x\rangle^{-1/2}w||^2.$$

Note that

$$\Im((A+\tilde{B}+\lambda^2)w, w) = \Im(-R_\infty(x,D)v + e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x), w) \tag{4.71}$$
$$\le \frac{1}{\varepsilon|\lambda|}||\langle x\rangle^{1/2}(R_\infty(x,D)v + e^{Mp}(x,D)\langle x\rangle^{-\mu})f(x)||^2$$
$$+\varepsilon|\lambda|||\langle x\rangle^{-1/2}w||^2.$$

Therefore we get from (4.65), (4.69), (4.70) and (4.71)

$$|\lambda|(\langle x\rangle^{-1/2}w||^2 \le |\lambda|\left\{(\langle x\rangle^{-1}e_0(D)w, e_0(D)w)\right. \tag{4.72}$$
$$\left.+(\langle x\rangle^{-1}e_1(D)w, e_1(D)w)\right\}$$
$$\le \frac{C}{|\lambda|}\left\{||\langle x\rangle^{1/2}e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x)||^2\right.$$
$$\left.+\ ||\langle x\rangle^{1/2}R_\infty(x,D)v||\right\},$$

for $|\lambda| \geq \lambda_0$. Denote $\eta(x,D) = -(A+\tilde{B}) + h^2$, where $h \in \mathbb{R}$ is a large parameter. Then $\eta(x,\xi) \in S(\langle\xi\rangle^2, g)$ satisfies $c_1\langle\xi\rangle^2 \leq |\eta(x,\xi)| \leq C_2\langle\xi\rangle^2$ and commutes with $A+\tilde{B}$. Therefore we can get analogously for any integer l and $|\lambda| \geq \lambda_0$

$$|\lambda|||\langle x\rangle^{-1/2}\eta(x,D)^l w||^2 \leq \frac{C}{|\lambda|}\{||\langle x\rangle^{1/2}\eta(x,D)^l R_\infty v||^2 \qquad (4.73)$$
$$+||\langle x\rangle^{1/2}\eta(x,D)^l e^{Mp}(x,D)\langle x\rangle^{-\mu} f(x)||^2\},$$

which implies

$$|\lambda|||(\langle x\rangle^{-1/2-\mu}\langle D\rangle^l e^{Mp}(x,D)u||^2 \qquad (4.74)$$
$$\leq \frac{C}{|\lambda|}\left\{||R'_\infty u||^2 + |\lambda|^{-1}||\langle x\rangle^{1/2-\mu+l_1}\langle D\rangle^l f(x)||^2\right\}.$$

For $e^{Mp}(x,D)$ we can find a parametrix $E(x,D)$ whose symbol is in $S\left(\langle x\rangle^{l_1}, \tilde{g}\right)$, $l_1 = C_1 M$ such that

$$E(x,D)e^{Mp}(x,D) = I + R'_\infty(x,D), \qquad (4.75)$$

where the symbol of R'_∞ is contained in $\bigcap_{l>0} S(\langle x\rangle^{-l}\langle\xi\rangle^{-l}, g)$. Hence

$$\langle x\rangle^{-l_1} = \langle x\rangle^{-l_1}E(x,D)e^{Mp}(x,D) - \langle x\rangle^{-l_1}R_\infty(x,D). \qquad (4.76)$$

Therefore, noting that the symbol of $\langle x\rangle^{-l_1}E(x,D)$ is in $S(1,\tilde{g})$ we get from (4.74) with $l=0$

$$|\lambda|||\langle x\rangle^{-1/2-\mu}\langle x\rangle^{-l_1}u||^2 \qquad (4.77)$$
$$= |\lambda|||\langle x\rangle^{-1/2-\mu-l_1}E(x,D)e^{Mp}(x,D)u - \langle x\rangle^{-1/2-\mu-l_1}R_\infty(x,D)u||^2$$
$$\leq C|\lambda|(||\langle x\rangle^{-1/2-\mu}e^{Mp}(x,D)v||^2 + ||R_\infty(x,D)u||^2)$$
$$\leq C(||\langle x\rangle^{1/2}R'_\infty u||^2 + |\lambda|||R"_\infty(x,D)u||^2)$$
$$+|\lambda|^{-1}||\langle x\rangle^{1/2}e^{Mp}(x,D)\langle x\rangle^{-\mu}f(x)||^2.$$

Besides,

$$|\lambda|||R'_\infty(x,D))u||^2 = |\lambda|(R'_\infty(x,D))(e_0(D)^2 + e_1(D)^2)u, R'_\infty(x,D)u)$$
$$\leq C||\langle x\rangle^{-1/2-\mu}u||^2 + |\lambda|||\langle x\rangle^{-1/2-\mu}e_1(D))u||^2 \qquad (4.78)$$

and analogously to (4.69) we can see that

$$|\lambda|||\langle x\rangle^{-1/2-\mu}e_1(D)u||^2 \leq C(||\langle x\rangle^{-1/2-\mu}u||^2 + |\lambda|^{-1}||\langle x\rangle^{-1/2-\mu}e_1(D)f(x)||^2). \qquad (4.79)$$

Therefore we can get (4.51) with $l=0$ and $\hat{l} = 2l_1$ from (4.77)-(4.79), if $|\lambda| \geq \lambda_0$. For $l > 0$ we can get (4.51) considering $\eta'(x,D)^l u$ instead of u and using the property of $\eta'(x,D)^l = (-A+h^2)^{\frac{l}{2}}$ which commutes with $A+\lambda^2$. □

Remark 4.9. In the above proof the condition (1.3) is not necessary. If the operator A is a perturbation of Δ, this theorem is proved in Murata [7].

The differentiability of $u(\lambda)$ with respect to λ follows from the theorem below.

Theorem 4.10. *Let* $\lambda = \sigma \pm i\varepsilon, \varepsilon \geq 0$, $|\lambda| \geq 1, f \in C_0^\infty(\mathbb{R}^n), \mu > 1/2$ *and* u *be a solution of equation* (4.22). *Denote* $u^k = \partial_\lambda^k u$. *Then there is* $C_{kl\mu} > 0$ *which is independent of* λ *such that*

$$||u^k(\lambda)||_{W^{l,2}_{-\mu-1/2-k}} \leq C_{kl}|\lambda|^{-1-k}||f||_{W^{l,2}_{-\mu+1/2+\hat{l}+k}}, \tag{4.80}$$

for $k, l = 0, 1, \ldots$ *and for* $|\lambda| \geq 1$, *where* $\hat{l}$ *is given in Theorem* 4.8.

It follows from Theorem 4.8 that (4.80) for $k = 0$ holds. The proof of (4.80) for $k \geq 1$ can be found in [3] and [4].

Theorem 3.2 is a direct result of Theorem 4.6 and Theorem 4.10. In fact, $w^1_\pm(x,\xi) = R((|\xi| \pm i0)^2)\{\sum_{l,k=1}^n \{(a_{lk}(x) - \delta_{lk})\xi_l\xi_k + \partial_{x_l} a_{lk}(x)\xi_k\}$ satisfies (3.4) from (4.40) and (4.80).

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On Weakly Pseudoconcave CR Manifolds

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1. Introduction

In [6] a weaker assumption of pseudoconcavity for abstract CR manifolds M of type (n, k) was introduced. The investigation in [6] concerned mostly the properties of local and global CR functions in M, while CR-meromorphic functions were considered in [7]. In this paper we investigate the consequences of this weaker assumption of pseudoconcavity, together with finite kind, for the top degree cohomology of the tangential Cauchy-Riemann complex. In particular we prove a result similar to [9] for real analytic CR manifolds which are of finite type and weakly pseudoconcave.

2. Preliminaries

An abstract smooth CR manifold $\mathbf{M}$ of type (n, k) is a triple (M, HM, J) consisting of a smooth paracompact manifold M of dimension $2n+k$, a smooth subbundle HM of TM of rank $2n$, and a smooth formally integrable complex structure J on the fibers of HM.

The formal integrability means that we have:

$$\left[\mathcal{C}^\infty(M, T^{0,1}M), \mathcal{C}^\infty(M, T^{0,1}M)\right] \subset \mathcal{C}^\infty(M, T^{0,1}M)\,, \tag{2.1}$$

where $T^{0,1}M$ is the complex subbundle of the complexification $\mathbb{C}HM$ of HM, which corresponds to the $(-i)$ eigenspace of J:

$$T^{0,1}M = \{X + iJX \,\big|\, X \in HM\}\,. \tag{2.2}$$

Next we define ${T^*}^{1,0}M$ as the annihilator of $T^{0,1}M$ in the complexified cotangent bundle $\mathbb{C}T^*M$. The integrability condition (2.1) is equivalent to the fact that the ideal $\mathcal{I}$ in the exterior algebra of smooth exterior differential forms generated by $\mathcal{C}^\infty(M, T^{*1,0}M)$ is d-closed: $d\mathcal{I} \subset \mathcal{I}$. This implies that also its powers $\mathcal{I}^p$ are d-closed ideals and we can consider for each p $(0 \leq p \leq n+k)$ a quotient complex of the deRham complex: $\bar{\partial}_M : \mathcal{I}^p \,/\mathcal{I}^{p+1} \to \mathcal{I}^p \,/\mathcal{I}^{p+1}$. These are complexes of smooth fiber bundles and first order partial differential operators, known as the *tangential Cauchy-Riemann complexes*:

$$\begin{aligned} 0 \to \mathcal{C}^\infty(M, Q^{p,0}M) \xrightarrow{\bar{\partial}_M} \mathcal{C}^\infty(M, Q^{p,1}M) &\xrightarrow{\bar{\partial}_M} \mathcal{C}^\infty(M, Q^{p,2}M) \\ \cdots \qquad &\xrightarrow{\bar{\partial}_M} \mathcal{C}^\infty(M, Q^{p,n}M) \to 0 \end{aligned} \tag{2.3}$$

where $\mathcal{C}^\infty(M, Q^{p,j}M) \simeq \mathcal{I}^p \cap \mathcal{C}^\infty(M, \Lambda^{p+j}\mathbb{C}T^*M) \,/\, \big(\mathcal{I}^{p+1} \cap \mathcal{C}^\infty(M, \Lambda^{p+j}\mathbb{C}T^*M)\big)$. We refer the reader to [3] and [4] for more details.

We define a sequence of $\mathbb{R}$-linear subspaces of the vector space of all global $\mathcal{C}^\infty$ real vector fields $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \mathcal{D}_3 \subset \cdots$ on M by:

$$\mathcal{D}_1 = \mathcal{C}^\infty(M, HM), \ldots, \; \mathcal{D}_j = \mathcal{D}_{j-1} + [\mathcal{D}_1, \mathcal{D}_{j-1}] \quad \text{for } j \geq 2\,. \tag{2.4}$$

For $x \in M$ we denote by $\left(\mathcal{D}_j\right)_x$ the real subspace of T_xM

$$\left(\mathcal{D}_j\right)_x \;=\; \left\{ X_x \,\middle|\, X \in \mathcal{D}_j \right\}\,. \tag{2.5}$$

If $\bigcup_j \left(\mathcal{D}_j\right)_x = T_xM$ we say that M has *finite kind* [1] at x and the number $\mu(x) = \inf\{j \mid \left(\mathcal{D}_j\right)_x = T_xM\}$ is the *kind* of M at x; the function $x \to \mu(x)$ is upper semi-continuous in M. The set of points where M has finite kind is open in M.

The *characteristic bundle* H^0M is defined to be the annihilator of HM in T^*M. Its purpose it to parametrize the Levi form: recall that the *Levi form* of M at x is defined for $\xi \in H^0_xM$ and $X \in H_xM$ by

$$\mathcal{L}_\xi(X) \;=\; d\tilde{\xi}(X, JX) = \langle \xi, [J\tilde{X}, \tilde{X}] \rangle\,, \tag{2.6}$$

where $\tilde{\xi} \in \mathcal{C}^\infty(M, H^0M)$ and $\tilde{X} \in \mathcal{C}^\infty(M, HM)$ are smooth extensions of ξ and X. For each fixed ξ it is a Hermitian quadratic form for the complex structure J_x on H_xM.

We say that a Riemannian metric h on M is partially Hermitian if its restriction to HM is Hermitian: this means that for all $x \in M$ the complex-valued form

$$H_xM \times H_xM \ni (X, Y) \to \tilde{h}(X, Y) = h(X, Y) \,+\, i\,h(X, J_xY) \in \mathbb{C}$$

is $\mathbb{C}$-linear in X and anti-$\mathbb{C}$-linear in Y for the complex structure J_x of H_xM.

Let $X_1, \ldots, X_n$ be an orthonormal basis of H_xM for the Hermitian scalar product defined by h_x. Then for each $\xi \in H^0_xM$ we can define the *trace* of $\mathcal{L}_\xi$ with respect to h by:

$$\mathrm{tr_h}(\mathcal{L}_\xi) = \sum_{\mathrm{j=1}}^{\mathrm{n}} \mathcal{L}_\xi(\mathrm{X_j})\,. \tag{2.7}$$

[1] Here we follow the terminology of Tanaka [15, 16]; this is often called finite type in the sense of Bloom-Graham.

The definition does not depend on the choice of the orthonormal basis, but only on the Hermitian metric $h|_{HM}$.

Definition 2.1. We say that M is *weakly pseudoconcave* if there exists a smooth partially Hermitian metric h in M such that

$$\operatorname{tr}_h(\mathcal{L}_\xi) = 0 \qquad \forall \xi \in H^0 M\,. \tag{2.8}$$

We recall the criterion (see [5, 6]):

Proposition 2.2. *Let M be a CR manifold of type (n,k). If $\mathcal{D}_2$ is a distribution of constant rank, then M is weakly pseudoconcave if and only if the following condition is satisfied:*

(∗) *For every $\xi \in H^0M$ either $\mathcal{L}_\xi = 0$, or $\mathcal{L}_\xi$ has at least one positive and one negative eigenvalue.*

Let $X_1, \dots, X_n \in \mathcal{C}^\infty(U, HM)$ define, for each point x of an open subset U of M, an orthonormal basis for $h_x\,|_{H_xM}$. Set:

$$L_j = X_j - iJX_j\,,\ \bar{L}_j = X_j + iJX_j \quad \text{for} \quad \mathrm{j} = 1, \dots, \mathrm{n}\,. \tag{2.9}$$

Then we obtain:

Lemma 2.3. *If* (2.8) *is valid, then*

$$\sum_{j=1}^{n} [L_j, \bar{L}_j] \in \mathcal{C}^\infty(U, \mathbb{C}HM)\,. \tag{2.10}$$

3. Finiteness and vanishing theorems

Let M be a weakly pseudoconcave CR manifold of type (n,k). Fix a smooth partially Hermitian metric h on M.

Let $x_0 \in M$ and let $X_1, \dots, X_n$ be an orthonormal basis for $\tilde{h}$ in a neighborhood U of x_0 in M; let $T_1, \dots, T_k$ be real vector fields, that we can assume to be defined in the same neighborhood U of x_0, such that

$$X_1, \dots, X_n\,, JX_1, \dots, JX_n\,, T_1, \dots, T_k$$

is an orthonormal basis of T_xM for h for every $x \in U$. Denote by

$$\Xi^1, \dots, \Xi^n, \Upsilon^1, \dots, \Upsilon^n, \xi^1, \dots, \xi^k$$

the dual basis in T^*U. We set $\omega^j = \Xi^j + i\Upsilon^j$, $\bar{\omega}^j = \Xi^j - i\Upsilon^j$ for $j = 1, \dots, n$, $\omega^j = \xi^{j-n}$ for $n+1 \le j \le n+k$. In this way, $\omega^1, \dots, \omega^{n+k}$ generate $T^{*1,0}_{x}M$ at each point $x \in M$.

An element $\alpha \in \mathcal{Q}^{p,q}(U)$ can be uniquely represented as a sum:

$$\alpha = \sum_{\substack{1 \le i_1 < \cdots < i_p \le n+k \\ 1 \le j_1 < \cdots < j_q \le n}} \alpha_{i_1,\dots,i_p;j_1,\dots,j_q} \omega^{i_1} \wedge \cdots \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q}\,. \tag{3.1}$$

We have:

$$\bar{\partial}_M \alpha = \sum_{\substack{1\leq i_1<\cdots<i_p\leq n+k \\ 1\leq j_1<\cdots<j_q\leq n}} \bar{\partial}_M \alpha_{i_1,\dots,i_p;j_1,\dots,j_q} \wedge \omega^{i_1} \wedge \cdots \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} + \cdots , \tag{3.2}$$

where "$\cdots$" stands for terms that are $\mathbb{C}$-linear in the coefficients $\alpha_{i_1,\dots,i_p;j_1,\dots,j_q}$ and $\bar{\partial}_M$ operates on smooth functions on M by:

$$\bar{\partial}_M v = \sum_{j=1}^{n} \bar{L}_j(v)\, \omega^j , \tag{3.3}$$

the operators $\bar{L}_j$ being defined as in (2.9). We can consider on the fibers of $\mathcal{Q}^{p,q}$ the Hermitian scalar product for which the forms:

$$\omega^{i_1} \wedge \cdots \omega^{i_p} \wedge \bar{\omega}^{j_1} \wedge \cdots \wedge \bar{\omega}^{j_q} , \; 1 \leq i_1 < \cdots < i_p \leq n+k, \; 1 \leq j_1 < \cdots < j_q \leq n$$

are an orthonormal basis. Using the Lebesgue measure associated to Riemannian metric h and the Hermitian scalar products on the fibers, we can define the *formal adjoint*

$$\begin{cases} \mathfrak{d}_M : \mathcal{Q}^{p,q+1}(M) \to \mathcal{Q}^{p,q}(M) \;\; q \geq 0, \; 0 \leq p \leq n+k \\ \mathfrak{d}_M = 0 \quad \text{on} \quad \mathcal{Q}^{p,0}(M) \quad \text{for} \quad 0 \leq p \leq n+k , \end{cases} \tag{3.4}$$

by requiring that for all $\alpha \in \mathcal{Q}^{p,q+1}(M)$, $\beta \in \mathcal{Q}^{p,q}(M)$ with a compact $\operatorname{supp}(\alpha) \cap \operatorname{supp}(\beta) \Subset M$:

$$\int (\alpha|\bar{\partial}_M \beta) d\lambda_h = \int (\mathfrak{d}_M \alpha|\beta) d\lambda_h . \tag{3.5}$$

We obtain the *adjoint Cauchy-Riemann complexes*, which are still complexes of first order partial differential operators:

$$\begin{aligned} 0 \leftarrow \mathcal{C}^\infty(M, Q^{p,0}M) \xleftarrow{\mathfrak{d}_M} \mathcal{C}^\infty(M, Q^{p,1}M) \xleftarrow{\mathfrak{d}_M} \mathcal{C}^\infty(M, Q^{p,2}M) \\ \dots \qquad \xleftarrow{\mathfrak{d}_M} \mathcal{C}^\infty(M, Q^{p,n}M) \leftarrow 0. \end{aligned} \tag{3.6}$$

Using (3.6), for each fixed $0 \leq p \leq n+k$ and $0 \leq q \leq n$ we can define a *Laplace operator* of (2.3) by:

$$\square_M : \mathcal{C}^\infty(M, Q^{p,q}M) \ni \alpha \to (\bar{\partial}_M \mathfrak{d}_M + \mathfrak{d}_M \bar{\partial}_M)\alpha \in \mathcal{C}^\infty(M, Q^{p,q}M) . \tag{3.7}$$

We note that the $\square_M$ operators are all second order and formally self-adjoint.

We recall from [6] that we have the following:

Proposition 3.1. *Assume that M is weakly pseudoconcave and fix a smooth partially Hermitian metric h on M for which (2.8) holds. Let $L_j = X_j - iJX_j$ $(1 \leq j \leq n)$ for an orthonormal basis $X_1, \dots, X_n$ of $\tilde{h}$ in an open subset U of M. Then, for $q = 0$ and $q = n$ the operators $\square_M$, in the trivialization of $\mathcal{Q}^{p,q}(U)$ given by (3.1),* *have an expression of the form:*

$$\Box_M = -\Big(\frac{1}{2}\sum_{j=1}^{n}\big(L_j\,\bar{L}_j + \bar{L}_j\,L_j + c_jL_j + \bar{c}_j\bar{L}_j\big)\Big)\cdot I + A(x) \tag{3.8}$$

where I is the identity and A a smooth $\binom{n+k}{p}\times\binom{n+k}{p}$ matrix of scalar functions.

It is worth to point out that, while the choice of a special smooth partially Hermitian metric h satisfying (2.8) is useful in some of the proofs, the results that we shall describe in the following do not depend on the choice of any special smooth partially Hermitian metric. In particular in the following we shall not require in the statements that h satisfies (2.8), although we shall need to fix some h to define the adjoint tangential Cauchy-Riemann complexes and the operators $\Box_M$.

With a proof similar with that of [6], Theorem 4.2, using [2] and [8], we obtain:

Theorem 3.2. *Assume that M is weakly pseudoconcave and of finite kind $\leq \delta^{-1} < +\infty$ at every point, and let h be a smooth partially Hermitian metric on M. Then for every $0 \leq p \leq n+k$ and every open $U \Subset M$ there exist real constants $c_0 > 0$ and C_1 such that*

$$\begin{aligned} &\|\bar{\partial}_M u\|_0^2 \geq c_0\|u\|_\delta^2 - C_1\|u\|_0^2 \quad \forall u \in \mathcal{Q}^{p,0}(M) \quad \textit{with} \quad \operatorname{supp}(u) \Subset U\,, \\ &\|\mathfrak{d}_M v\|_0^2 \geq c_0\|v\|_\delta^2 - C_1\|v\|_0^2 \quad \forall v \in \mathcal{Q}^{p,n}(M) \quad \textit{with} \quad \operatorname{supp}(v) \Subset U\,. \end{aligned} \tag{3.9}$$

The operators $\bar{\partial}_M : \mathcal{Q}^{p,0}(M) \longrightarrow \mathcal{Q}^{p,1}(M)$ and $\mathfrak{d}_M : \mathcal{Q}^{p,n}(M) \longrightarrow \mathcal{Q}^{p,n-1}(M)$ are hypo-elliptic with a loss of $(1-\delta)$ derivatives. This means that, for all $s \geq 0$:

$$u \in L^2_{\text{loc}}(M, Q^{p,0}M) \quad \textit{and} \quad \bar{\partial}_M u \in W^s_{\text{loc}}(M, Q^{p,1}M) \Longrightarrow u \in W^{s+\delta}_{\text{loc}}(M, Q^{p,0}M)\,,$$

$$v \in L^2_{\text{loc}}(M, Q^{p,n}M) \quad \textit{and} \quad \mathfrak{d}_M v \in W^s_{\text{loc}}(MQ^{p,n-1}M) \Longrightarrow v \in W^{s+\delta}_{\text{loc}}(M, Q^{p,n}M)\,.$$

Here $\|\cdot\|_0$ and $\|\cdot\|_\delta$ are respectively the L^2 and the Sobolev δ-norms, that can be computed after having fixed the Riemannian metric on M and the Hermitian metrics on the fibers of $Q^{p,q}M$. We use the notation $L^2(M, Q^{p,q}M)$ to indicate the space of L^2 sections of $Q^{p,q}M$ and $W^s(M, Q^{p,q}M)$ for those that have fractional L^2-derivatives up to order s. Note that different choices of h give equivalent L^2 and W^s norms on compact subsets of M. Arguing as in [6], §5 and [1] we also obtain:

Theorem 3.3. *Assume that M is connected, weakly pseudoconcave and minimal. Let $u \in L^2_{\text{loc}} \otimes \mathcal{Q}^{p,0}(M)$ and assume that there is a nonnegative real-valued function $\kappa \in L^\infty_{\text{loc}}(M)$ such that:*

$$|\partial_M u(x)| \leq \kappa(x)\,|u(x)| \quad \textit{a.e. in } M\,. \tag{3.10}$$

Then, if $u = 0$ a.e. in an open subset of M, then $u = 0$ a.e. in M.

Likewise, let $v \in L^2_{\text{loc}} \otimes \mathcal{Q}^{p,n}(M)$ and assume that there is a nonnegative real valued function $\kappa \in L^\infty_{\text{loc}}(M)$ such that:

$$|\mathfrak{d}_M v(x)| \leq \kappa(x)\,|v(x)| \quad \textit{a.e. in } M\,. \tag{3.11}$$

Then, if $v = 0$ a.e. in an open subset of M, then $v = 0$ a.e. in M.

We recall that *minimal* means that for each point $x_0 \in M$ and every open neighborhood U of x_0 in M, the set of points $x \in U$, for which there exists a piece-wise $\mathcal{C}^1$ path $s : [0,1] \to U$ with $s(0) = x_0$, $s(1) = 1$ and $\dot{s}(t) \in H_{s(t)}M$ for almost all $t \in [0,1]$, is an open neighborhood of x_0 in M. This condition is more general than finite kind (see [17, 18]).

As a consequence of the regularity and the weak unique continuation theorem we obtain:

Corollary 3.4. *Assume that M is weakly pseudoconcave and of finite kind $\leq \delta^{-1} < +\infty$ at every point, and let h be a smooth partially Hermitian metric on M. Then for every $0 \leq p \leq n+k$ and every open $U \Subset M$, such that U does not contain any non-empty compact component of M, and $0 \leq \delta' < \delta$ there exist real constants $c(U) > 0$ such that*

$$\begin{aligned} \|\bar{\partial}_M u\|_0^2 &\geq c(U)\|u\|_{\delta'}^2 \quad \forall u \in \mathcal{Q}^{p,0}(M) \quad \textit{with} \quad \operatorname{supp}(u) \Subset U\,, \\ \|\mathfrak{d}_M v\|_0^2 &\geq c(U)\|v\|_{\delta'}^2 \quad \forall v \in \mathcal{Q}^{p,n}(M) \quad \textit{with} \quad \operatorname{supp}(v) \Subset U\,. \end{aligned} \tag{3.12}$$

Proof. We shall prove for instance the second of these inequalities. Assume by contradiction that, for some fixed p, there is for every integer $m > 0$ a $v_m \in \mathcal{Q}^{p,n}(M)$ with

$$\operatorname{supp}(v) \Subset U \quad \text{and} \quad m^2\|\mathfrak{d}_M v_m\|_0^2 < \|v_m\|_{\delta'}^2.$$

By rescaling we can assume that $\|v_m\|_{\delta'}^2 = 1$ for all m. Then, by Theorem 3.2 the norm $\|v_m\|_\delta^2$ is uniformly bounded with respect to m and by passing to a subsequence we can assume that v_m strongly converges in $L^2(M)$ to an $L^2(M)$-function v_∞ which satisfies $\|v_\infty\|_{\delta'} = 1$, so that $v_\infty \neq 0$. But we also have $\mathfrak{d}_M v_\infty = 0$ and $\operatorname{supp}(v_\infty) \subset \overline{U} \Subset M$, contradicting the weak unique continuation principle. ☐

We also obtain:

Corollary 3.5. *Assume that M is weakly pseudoconcave and of finite kind $\leq \delta^{-1} < +\infty$ at every point, and let h be a smooth partially Hermitian metric on M. Then for every $0 \leq p \leq n+k$ and every open $U \Subset M$ there exist real constants $c_0' > 0$ C_1' such that*

$$\begin{aligned} \|\Box_M u\|_0^2 &\geq c_0'\|u\|_{2\delta}^2 - C_1'\|u\|_0^2 \quad \forall u \in \mathcal{Q}^{p,0}(M) \quad \textit{with} \quad \operatorname{supp}(u) \Subset U\,, \\ \|\Box_M v\|_0^2 &\geq c_0'\|v\|_{2\delta'}^2 - C_1'\|v\|_0^2 \quad \forall v \in \mathcal{Q}^{p,n}(M) \quad \textit{with} \quad \operatorname{supp}(v) \Subset U\,. \end{aligned} \tag{3.13}$$

Moreover, the operators $\Box_M$ in $\mathcal{Q}^{p,0}(M)$ and in $\mathcal{Q}^{p,n}(M)$ are hypo-elliptic for every $0 \leq p \leq n+k$ with a loss of $(2-2\delta)$ derivatives.

Proof. Take $v \in \mathcal{Q}^{p,n}$ with $\operatorname{supp}(v) \Subset U$. By the hypo-ellipticity of $\mathfrak{d}_M$, there are constants $k_0 > 0$, k_1, k_2 such that:

$$k_0\|v\|_{2\delta}^2 - k_1\|v\|_0^2 \leq \|\mathfrak{d}_M v\|_\delta^2 \leq (\Box_M v \,|\, v)_\delta + k_2\|\mathfrak{d}_M v\|_0\|v\|_{2\delta}$$

from which we obtain at once the second line of (3.13). The first line of (3.13) is obtained in a similar way. Also the hypo-ellipticity with the loss of $(2-2\delta)$ derivatives is deduced in a similar way from that of $\bar{\partial}_M$ and $\mathfrak{d}_M$. ☐

Theorem 3.6. *Assume that M is weakly pseudoconcave and of finite kind. Then, if M is compact, all cohomology groups $H^n(M, \mathcal{Q}^{p,*}, \bar{\partial}_M)$ are finite-dimensional.*

Proof. Consider the operator $\Box_M : \mathcal{Q}^{p,n}(M) \to \mathcal{Q}^{p,n}(M)$. Being hypo-elliptic, it defines a compact linear map $L^2(M, Q^{p,n}M) \to L^2(M, Q^{p,n}M)$, and therefore has a finite kernel and cokernel. Denote by $\mathcal{N}$ its kernel. Then $\mathcal{N} \subset \mathcal{Q}^{p,n}(M)$ and for all $f \in \mathcal{N}^\perp \cap \mathcal{Q}^{p,n}(M)$ there is $v \in \mathcal{Q}^{p,n}(M)$ such that $\Box_M v = f$. But this implies that $w = \mathfrak{d}_M v$ solves $\bar{\partial}_M w = f$. This shows that the inclusion $\mathcal{N} \hookrightarrow \mathcal{Q}^{p,n}(M)$ induces a surjective map $\mathcal{N} \to H^n(M, \mathcal{Q}^{p,*}, \bar{\partial}_M)$. This completes the proof. $\Box$

Lemma 3.7. *Assume that M is weakly pseudoconcave and of finite kind. Then, for every open subset U of M with $U \Subset M$, such that U does not contain any compact component of M, the natural map induced by restrictions: $H^n(M, \mathcal{Q}^{p,*}, \bar{\partial}_M) \to H^n(U, \mathcal{Q}^{p,*}, \bar{\partial}_M)$ has zero image for all $0 \le p \le n+k$.*

Proof. We use Corollary 3.4. We obtain, with $c = c(U) > 0$:

$$c\,\|v\|_0^2 \le \|\mathfrak{d}_M v\|^2 = (\Box_M v|v)_0 \le \|\Box_M v\|_0\,\|v\|_0 \quad \forall v \in \mathcal{Q}^{p,n}(M) \text{ with } \operatorname{supp}(v) \Subset U\,.$$

Hence

$$c\,\|v\|_0 \le \|\Box_M v\|_0 \quad \forall v \in \mathcal{Q}^{p,n}(M) \text{ with } \operatorname{supp}(v) \Subset U\,.$$

Let $f \in \mathcal{Q}^{p,n}(M)$. Then we have:

$$(f|v)_0 \le \|f|_U\|_0\,\|v\|_0 \le c^{-1}\,\|f|_U\|_0\,\|\Box_M v\|_0 \quad \forall v \in \mathcal{Q}^{p,n}(M) \text{ with } \operatorname{supp}(v) \Subset U\,.$$

This shows that $\Box_M v \to (f|v)$ is linear and continuous on the linear subspace $\{\Box_M v \,|\, v \in \mathcal{Q}^{p,n}(M) \text{ with } \operatorname{supp}(v) \Subset U\} \subset L^2(U, Q^{p,n}M)$. By Hahn-Banach Theorem there is $u \in L^2(U, Q^{p,n}M)$ such that:

$$(u|\Box_M v) = (f, v) \quad \forall v \in \mathcal{Q}^{p,n}(M) \text{ with } \operatorname{supp}(v) \Subset U\,.$$

This means that $\Box_M u = f$ a.e. in U, and since $\Box_M$ is hypo-elliptic, it follows that $u \in \mathcal{Q}^{p,n}(U)$. Thus $w = \mathfrak{d}_M u \in \mathcal{Q}^{p,n-1}(U)$ is a smooth solution of $\bar{\partial}_M w = f$. $\Box$

Theorem 3.8. *Assume that M is a real analytic CR manifold of type (n, k) which is weakly pseudoconcave and of finite kind. Assume that on M it is possible to define a real analytic partially Hermitian metric. If no connected component of M is compact, then $H^n(M, \mathcal{Q}^{p,*}, \bar{\partial}_M) = 0$ for all $0 \le p \le n+k$.*

We begin by proving the following:

Lemma 3.9. *Under the assumptions of Theorem 3.8, for every $0 \le p \le n+k$, the operator $\Box_M : \mathcal{Q}^{p,n}(M) \to \mathcal{Q}^{p,n}(M)$, constructed by using a real analytic partially Hermitian metric, has the weak unique continuation property.*

Proof. We can assume that M is connected. Let $u \in \mathcal{Q}^{p,n}(M)$ be a solution of $\Box_M u = 0$, and assume by contradiction that $\operatorname{supp}(u)$ is neither $\emptyset$, nor M. Then (see [1], Theorem 2.1) we can find a smooth function $\phi : M \to \mathbb{R}$ with $\phi(x) \le 0$ for $x \in \operatorname{supp}(u)$, and $\phi(x_0) = 0$, $d\phi(x_0) \notin H_{x_0}M$ for some $x_0 \in \operatorname{supp}(u)$. Then

$d\phi(x_0)$ is non characteristic for $\Box_M$, which is a partial differential operator with real analytic coefficients. Since u satisfies $\Box_M u = 0$ in M with zero Cauchy data on $\phi = 0$, by Holmgren's uniqueness theorem it follows that u equals 0 in a neighborhood of x_0, contradicting that $x_0 \in \operatorname{supp}(u)$. $\Box$

Lemma 3.10. *Under the assumptions of Theorem* 3.8, *if U is a relatively compact open subset of M, then every $u \in \mathcal{E}'(U, Q^{p,n})$ such that*

$$\langle u, \bar{v}\rangle = 0 \quad \forall u \in \mathcal{Q}^{p,n}(U) \text{ with } \Box_M v = 0 \tag{3.14}$$

is of the form $u = \Box_M w$ for some $w \in \mathcal{E}'(U, Q^{p,n})$.

Here the pairing $\langle u, \bar{v}\rangle$ is the extension of the L^2 scalar product in $L^2(U, Q^{p,n}M)$. Also we note that for $s \geq 0$ and U open relatively compact in M and with a smooth boundary, the space $W^{-s}(U, Q^{p,q}M)$ can be considered as the subspace of $u \in W^{-s}(M, Q^{p,q}M)$ with $\operatorname{supp}(u) \subset \bar{U}$. This follows indeed by duality from the fact that the restriction map $W^s(M, Q^{p,q}M) \to W^s(U, Q^{p,q}M)$ is continuous and onto.

Proof of Lemma 3.10. Let U' be a relatively compact open neighborhood of $\bar{U}$ in M, with a smooth boundary and such that $U' \setminus U$ has no compact connected component. Let $u \in \mathcal{E}'(U, Q^{p,n})$ satisfy (3.14). Let $s \geq 2$ be a positive real number such that $u \in W^{-s}(U, Q^{p,n}M)$. Since all distribution solutions of $\Box_M v = 0$ in U' are smooth, we have in particular:

$$\langle u, \bar{v}\rangle = 0 \quad \forall u \in W^s(U', Q^{p,n}M) \text{ with } \Box_{\mathrm{M}}\mathrm{v} = 0\,.$$

Thus u is orthogonal to the kernel of the map

$$A : W^s(U', Q^{p,n}M) \ni v \to \Box_M v \in W^{s-2}(U', Q^{p,n}M)$$

and therefore belongs to the closure of the image of the adjoint map

$$A^* : W^{2-s}(U', Q^{p,n}M) \ni w \to \Box_M w \in W^{-s}(U', Q^{p,n}M).$$

Hence there exists a sequence $\{w_m\} \subset W^{2-s}(U', Q^{p,n}M)$ such that $\Box_M w_m \to u$ in $W^{-s}(U', Q^{p,n}M)$. By the hypoellipticity of $\Box_M$, and the weak unique continuation property, we have, for some positive constant $c' > 0$, that

$$\|\Box_M w_m\|_{-s} \geq c'\|w_m\|_{2\delta-s-2} \quad \text{for all } m\,.$$

This follows indeed by an argument similar to the proof of Corollary 3.4. Thus a subsequence of $\{w_m\}$ converges to a distribution $w \in \mathcal{E}'(M, Q^{p,n})$ with $\operatorname{supp}(w) \subset \bar{U}'$ and $\Box_M w = u$. We note that $\Box_M(w) = 0$ in $M \setminus \operatorname{supp}(u)$. Since $\bar{U}'$ has a neighborhood U'' that is relatively compact in M and is such that $U'' \setminus U$ has no compact connected components, from the weak unique continuation proved in the previous lemma, we deduce that $\operatorname{supp}(w) \subset U$. This completes the proof. $\Box$

Proof of Theorem 3.8. We fix a real analytic partially Hermitian metric h in M, so that $\Box_M$ has real analytic coefficients. Let $\{U_m\}_{m\geq 0}$ be a sequence of open subsets of M with

$$U_m \Subset U_{m+1}, \quad U_{m+1} \setminus U_m \text{ has no compact component}, \quad \bigcup U_m = M\,.$$

Let $f \in \mathcal{Q}^{p,n}(M)$ be given. By Lemma 3.7 for each m we can find $u_m \in \mathcal{Q}^{p,n}(U_m)$ such that

$$\Box_M u_m = f \text{ in } U_m\,.$$

Denote by $\mathcal{N}(U)$ the set of functions v in $\mathcal{Q}^{p,n}(U)$ that satisfy the homogeneous equation: $\Box_M v = 0$ in U.

Consider for each m the restriction map: $r_{U_m}^{U_{m+1}} : \mathcal{N}(U_{m+1}) \to \mathcal{N}(U_m)$. We claim that this map has a dense image. This is indeed equivalent to the fact that the adjoint map is injective. Considering the standard Fréchet-Schwartz topology in the spaces $\mathcal{N}(U_*)$, it suffices to prove that:

If $T \in \mathcal{E}'(U_{m+1}, Q^{p,n})$ *and* $\operatorname{supp}(\Box_M T) \Subset U_m$, *then* $\operatorname{supp}(T) \subset U_m$.

This follows from Lemma 3.9, because $U_{m+1} \setminus U_m$ has no compact component.

Now we claim that there is a sequence $\{v_m\}$ with $v_m \in \mathcal{N}(U_m)$ such that, for $w_m = u_m - v_m$, we have:

$$\|(w_{m+1} - w_m)|_{U_{m-1}}\|_0^2 < 2^{-m} \tag{3.15}$$

for all $m \geq 2$. Indeed, we take $v_0 = 0$, $v_1 = 0$, and assume that we have already chosen $v_m \in \mathcal{N}(U_m)$ for $1 \leq m \leq m_0$ in such a way that (3.15) is valid for $2 \leq m \leq m_0$. Then we observe that the restriction of $(u_{m_0+1} - w_{m_0})$ to U_{m_0} is an element of $\mathcal{N}(U_{m_0})$. Because the restriction map $\mathcal{N}(U_{m_0+1}) \to \mathcal{N}(U_{m_0})$ has a dense image, we can find $v_{m_0+1} \in \mathcal{N}(U_{m_0+1})$ such that $\|(u_{m_0+1} - w_{m_0} - v_{m_0+1})|_{U_{m_0-1}}\|^2 < 2^{-m_0-1}$. This is (3.15) for $w_{m_0+1} = (u_{m_0+1} - v_{m_0+1})$ and $m = m_0 + 1$. Thus by recurrence we obtain the sequence $\{w_m\}$ with the desired properties. Finally we define u in M by setting:

$$u \,=\, w_m \,+\, \sum_{h=m}^{\infty} (w_{m+1} - w_m) \quad \text{in} \quad U_m\,.$$

Clearly the series converges and yields a function $u \in L^2_{\text{loc}}(M, Q^{p,n})$. This u satisfies $\Box_M u = f$. Then u is smooth because $\Box_M$ is hypo-elliptic and hence $g = \vartheta_M u \in \mathcal{Q}^{p,n}(M)$ and satisfies $\bar{\partial}_M g = f$. $\Box$

4. Examples

1. We consider the canonical isomorphism $\mathbb{C}^8 \simeq \mathbb{H}^4$, defined by requiring that the left product of $\jmath$ by a vector of the canonical basis $e_1, \ldots, e_8$ satisfies the multiplication rule:

$$\begin{aligned} \jmath \cdot e_{2h-1} &= e_{2h}, \\ \jmath \cdot e_{2h} &= -e_{2h-1}, \end{aligned}$$

for $h = 1, 2, 3, 4$.

Let $\tilde{M} = \mathfrak{Gr}_{4,8}$ be the Grassmannian of complex 4-planes of $\mathbb{C}^8$. It is a 16-dimensional compact complex manifold. Near the point $\langle e_1, e_2, e_3, e_4 \rangle$ of $\tilde{M}$ we can fix coordinates $z_1, \ldots, z_{12}, w_1, \ldots, w_4 \in \mathbb{C}$, to which we let correspond the 4-plane of $\mathbb{C}^8$ generated by the columns of the matrix:

$$\begin{pmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \\ z_3 & z_4 & z_9 & z_{10} \\ z_1 & z_2 & z_{11} & z_{12} \\ w_1 & w_3 & z_5 & z_7 \\ w_2 & w_4 & z_6 & z_8 \end{pmatrix}$$

We shall consider the non-compact CR sub-manifold M of $\tilde{M}$ consisting of those 4-planes ℓ_4 of $\mathbb{C}^8$ for which the intersection $\ell_4 \cap (\jmath \cdot \ell_4)$ has complex dimension 2. It is the orbit of the 4-plane $\mathfrak{o} = \langle e_1, e_2, e_3, e_5 \rangle$ under the action of the group $\mathbf{SL}(4, \mathbb{H})$ acting on the right on $\mathbb{C}^8 \simeq \mathbb{H}^4$. It is convenient to represent the vectors of the tangent space to M at the point $\mathfrak{o}$ by matrices of the Lie algebra $\mathfrak{sl}(4, \mathbb{H}) \subset \mathfrak{sl}(8, \mathbb{C})$, written in the ordered basis $e_1, e_2, e_3, e_5, e_4, e_6, e_7, e_8$. In this way the vectors of $H_{\mathfrak{o}}M$ are parametrized by those entries a_{ij} of the bottom left 4×4 sub-matrix (i.e., $5 \leq i \leq 8$, $1 \leq j \leq 4$) for which $\pm \bar{a}_{ij}$ equals some entry that does not belong to the same sub-matrix. We used z_i's for these coordinates and w_i's for those having their conjugation within the same bottom left 4×4 sub-matrix. Thus $H_{\mathfrak{o}}M$ is represented by:

$$\begin{pmatrix} 0 & 0 & & & & & & \\ 0 & 0 & & & & & & \\ \bar{z}_4 & -\bar{z}_3 & 0 & 0 & -\bar{z}_9 & -\bar{z}_{10} & & \\ \bar{z}_2 & -\bar{z}_1 & 0 & 0 & -\bar{z}_{12} & -\bar{z}_{11} & & \\ z_3 & z_4 & z_9 & z_{10} & 0 & 0 & & \\ z_1 & z_2 & z_{12} & z_{11} & 0 & 0 & & \\ w_1 & -\bar{w}_2 & z_5 & z_7 & -\bar{z}_6 & -\bar{z}_8 & 0 & 0 \\ w_2 & \bar{w}_1 & z_6 & z_8 & \bar{z}_5 & \bar{z}_7 & 0 & 0 \end{pmatrix}$$

This shows that M is a CR manifold of CR dimension 12 and CR codimension 4. Our representation can also be used to compute the Levi form of our $M^{12,4}$.

We can parametrize $H^0_{\mathfrak{o}}M$ by the real and imaginary parts of the two complex variables w_1 and w_2. Then the Levi form depends upon two complex variables α, β and we obtain:

$$\mathcal{L}_{\alpha,\beta} = \begin{pmatrix} & & & & & & -\beta & \alpha & & & & \\ & & & & & & \bar{\alpha} & \bar{\beta} & & & & \\ & & & & -\beta & \alpha & & & & & & \\ & & & & \bar{\alpha} & \bar{\beta} & & & & & & \\ & & -\bar{\beta} & \alpha & & & & & & & & \\ & & \bar{\alpha} & \beta & & & & & & & & \\ -\bar{\beta} & \alpha & & & & & & & & & & \\ \bar{\alpha} & \beta & & & & & & & & & & \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \\ & & & & & & & & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since M is homogeneous and the Levi form is invariant by CR isomorphisms, we conclude that for all $\xi \neq 0$ in H^0M the Levi form $\mathcal{L}_\xi$ has exactly 4 positive, 4 negative, and 4 zero eigenvalues.

Since M is homogeneous it is possible to define a real analytic partially Hermitian metric on M and thus we can conclude that

$$H^{12}(M, \mathcal{Q}^{p,*}, \bar{\partial}_M) = 0$$

for $0 \leq p \leq 16$.

The maximal compact subgroup of $\mathbf{SL}(4, \mathbb{H})$ is the group $\mathbf{Sp}(4)$ of quaternionic-unitary transformations. It can be identified to the intersection of $\mathbf{SL}(4, \mathbb{H})$, considered as a subgroup of $\mathbf{SL}(8, \mathbb{C})$, with the group $\mathbf{U}(8)$ of unitary transformations of $\mathbb{C}^8$.

Denoting by $(v \,|\, w)$ the Hermitian scalar product in $\mathbb{C}^8$, we note that we have the formula:

$$(\jmath \cdot v \,|\, \jmath \cdot w) = \overline{(v \,|\, w)}$$

for all $v, w \in \mathbb{C}^8$.

The orbit of $\mathfrak{o} = \langle e_1, e_2, e_3, e_5 \rangle$ by the action of $\mathbf{Sp}(4)$ is a compact CR manifold N, of type $(11, 4)$, contained in M:

$$N = \{\ell_2 \oplus r_2 \,|\, \ell_2, r_2 \in \mathfrak{Gr}_{2,8},\ \ell_2 \perp r_2,\ \ell_2 = \jmath \cdot \ell_2\ \ r_2 \perp \jmath \cdot r_2\}$$

where $\mathfrak{Gr}_{2,8}$ is the Grassmannian of complex 2-planes in $\mathbb{C}^8$.

This can be better checked by representing the tangent space $T_{\mathfrak{o}}N$ to N at $\mathfrak{o}$ by matrices of $\mathbf{Sp}(4)$:

$$\begin{pmatrix}
0 & 0 & -z_4 & -z_2 & -\bar{z}_3 & -\bar{z}_1 & -\bar{w}_1 & -\bar{w}_2 \\
0 & 0 & z_3 & z_1 & -\bar{z}_4 & -\bar{z}_2 & w_2 & w_1 \\
\bar{z}_4 & -\bar{z}_3 & 0 & 0 & -\bar{z}_9 & -\bar{z}_{10} & -\bar{z}_5 & -\bar{z}_6 \\
\bar{z}_2 & -\bar{z}_1 & 0 & 0 & -\bar{z}_{10} & -\bar{z}_{11} & -\bar{z}_7 & -\bar{z}_8 \\
z_3 & z_4 & z_9 & z_{10} & 0 & 0 & z_6 & -z_5 \\
z_1 & z_2 & z_{10} & z_{11} & 0 & 0 & z_8 & -z_7 \\
w_1 & -\bar{w}_2 & z_5 & z_7 & -\bar{z}_6 & -\bar{z}_8 & 0 & 0 \\
w_2 & \bar{w}_1 & z_6 & z_8 & \bar{z}_5 & \bar{z}_7 & 0 & 0
\end{pmatrix}$$

Thus the tangent space to N at $\langle e_1, e_2, e_3, e_5\rangle$ is characterized inside $T_{\mathfrak{o}}M$ by the *complex* linear equation $z_{10} = z_{12}$.

We can represent M in the form:

$$M = \{\ell_2 \oplus r_2 \,|\, \ell_2, r_2 \in \mathfrak{Gr}_{2,8},\ \ell_2 \perp r_2,\ \ell_2 = \jmath \cdot \ell_2\ \ r_2 \cap \jmath \cdot r_2 = \{0\}\}$$

and consider the natural map $\pi : M \ni \ell_2 \oplus r_2 \to r_2 \in \mathfrak{Gr}_{2,8}$. Given $\ell_4 = \ell_2 \oplus r_2 \in M$, fix any orthonormal basis v_1, v_2 of r_2 (with respect to the Hermitian scalar product in $\mathbb{C}^8$). Then $|(v_1 \,|\, \jmath \cdot v_2)|$ is independent of the choice of the orthonormal basis. Indeed, if $w_1 = \alpha\, v_1 + \beta\, v_2$, $w_2 = \gamma\, v_1 + \delta\, v_2$, is another orthonormal basis, we have:

$$\begin{aligned}
(w_1 \,|\, \jmath \cdot w_2) &= (\alpha\, v_1 + \beta\, v_2 \,|\, \bar{\gamma}\, \jmath \cdot v_1 \ +\ \bar{\delta}\, \jmath \cdot v_2) \\
&= \alpha \cdot \delta\, (v_1 \,|\, \jmath \cdot v_2)\ +\ \beta \cdot \gamma\, (v_2 \,|\, \jmath \cdot v_1) \\
&= \alpha \cdot \delta\, (v_1 \,|\, \jmath \cdot v_2)\ +\ \beta \cdot \gamma\, \overline{(\jmath \cdot v_2 \,|\, \jmath \cdot \jmath \cdot v_1)} \\
&= \alpha \cdot \delta\, (v_1 \,|\, \jmath \cdot v_2)\ -\ \beta \cdot \gamma\, \overline{(\jmath \cdot v_2 \,|\, v_1)} \\
&= (\alpha \cdot \delta - \beta \cdot \gamma)\, (v_1 \,|\, \jmath \cdot v_2)
\end{aligned}$$

which proves our contention because $|\alpha \cdot \delta - \beta \cdot \gamma| = 1$, being the determinant of a unitary transformation.

We observe that $\phi(\ell_4) = -\log(1 - |(v_1 \,|\, \jmath \cdot v_2)|^2)$ for an orthonormal basis v_1, v_2 of $\pi(\ell_4)$. In this way we obtain an exhausting function for M with $\bar{\partial}_M \phi \neq 0$ for $\ell_4 \notin N$.

It would be interesting to use this exhausting function to try to obtain weighted L^2 estimate to study the cohomology of the tangential Cauchy-Riemann complex of M.

2. We consider on $\mathbb{C}^6$ a Hermitian form K with 3 positive and 3 negative eigenvalues. We consider the 11-dimensional compact complex manifold $\tilde{M}$ consisting of the pairs (ℓ_1, ℓ_3) of a complex line ℓ_1 and a 3-dimensional plane with $\ell_1 \subset \ell_3$. We shall consider the CR submanifold of $\tilde{M}$ consisting of the pairs $\ell_1 \subset \ell_3$ with $v^* K v = 0$ for all $v \in \ell_3$. Near the point $(\langle\, e_1 \,\rangle, \langle\, e_1, e_2, e_3 \,\rangle)$ we can use complex local coordinates $z_1, \ldots, z_6$, $w_1, \ldots, w_4$, τ, so that ℓ_1 is generated by the first column and ℓ_3 by the columns of the matrix:

$$Z = \begin{pmatrix} 1 & & \\ w_1 & 1 & \\ w_2 & 0 & 1 \\ z_1 & z_2 & z_3 \\ \tau & w_3 & w_4 \end{pmatrix}.$$

With

$$K = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}$$

the submanifold M is described in local coordinates by:

$$Z^* K Z = 0\,.$$

One easily verifies that M satisfies the weak pseudoconcavity condition, but is of infinite kind at every point.

3. Consider in $\mathbb{C}^6$ the Hermitian symmetric matrix

$$K = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & 1 & & \\ & & 1 & & & \\ & 1 & & & & \\ 1 & & & & & \end{pmatrix}.$$

To a vector $z = (z^1, \ldots, z^{11})$ of $\mathbb{C}^{11}$ we associate the matrix:

$$A(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ z_1 & 1 & 0 & 0 \\ z_2 & 0 & 1 & 0 \\ z_3 & 0 & 0 & 1 \\ z_4 & z_5 & z_6 & z_7 \\ z_8 & z_9 & z_{10} & z_{11} \end{pmatrix}$$

and we denote by $v_1(z)$ its first column. Denote by M the smooth submanifold of $\mathbb{C}^{11}$ described by the equations:

$$v_1(z)^* K v_1(z) = 0\,, \quad \mathrm{rank}([\mathrm{A(z)}]^* \mathrm{KA(z)}) = 2\,.$$

Then M is a real analytic CR submanifold of $\mathbb{C}^{11}$ of type $(3, 8)$, which is weakly pseudoconcave and of finite kind. Actually its kind is 4 and there is a 3-dimensional subspace V_x of $H^0_x M$, for each $x \in M$, such that the Levi form $\mathcal{L}_\xi$ is zero for $\xi \in V_x$. Hence M is not pseudoconcave in the strong sense of [3] or [9]. Since M is homogeneous, it admits a real analytic partially Hermitian metric. We note that M is connected and is not compact. Hence $H^3(M, \mathcal{Q}^{p,*}, \bar\partial_M) = 0$ for all $0 \le p \le n+k$.

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A Note on Kohn's and Christ's Examples

Cesare Parenti and Alberto Parmeggiani

Abstract. We give here a family of second order examples tailored to those by Kohn and by Christ, which are $(C\quad)$ hypoelliptic and lose an arbitrarily large (fixed) number of derivatives.

1. Introduction

We start by fixing what is meant here by (C^∞) *hypoellipticity with loss of derivatives.*

Given a classical (properly supported) pseudodifferential operator (ψdo) A of order m on some open set $X \subset \mathbb{R}^n$, we say that: *A is (C^∞) hypoelliptic at a point $\rho \in T^*X \setminus 0$ with loss of $r(\geq 0)$ derivatives if for any given $u \in \mathcal{D}'(X)$ and for any given $s \in \mathbb{R}$*

$$Au \in H^s \text{ at } \rho \implies u \in H^{s+m-r} \text{ at } \rho,$$

where for a distribution $v \in \mathcal{D}'(X)$ to belong to H^t at *ρ means that $v = v' + v''$ with $v' \in H^t_{\mathrm{loc}}(X)$ and $\rho \notin \mathrm{WF}(v'')$.*

Furthermore, A is (C^∞) hypoelliptic at $z_0 \in X$ with loss of r derivatives if A is hypoelliptic at $(z_0, \zeta) \in T^*X \setminus 0$ with loss of r derivatives for all directions $\zeta \in T^*_{z_0}X \setminus \{0\}$.

In very recent papers Kohn [5] and Christ [2] have given (*inter alia*) examples of second order differential operators, sums of squares of two complex vector fields, that are hypoelliptic with an arbitrarily large (but fixed) loss of derivatives.

On the other hand, in another recent paper [6], we have given necessary and sufficient conditions for the hypoellipticity with loss of an arbitrary number r of derivatives, for a class of ψdo's with symplectic characteristics.
Motivated by Kohn's and Christ's examples, we show here how the machinery developed in [6] allows us to construct another family of examples of second order operators that are hypoelliptic with loss of as many derivatives as we wish. The proof of the hypoellipticity is a straightforward consequence of the general theory developed in [6], theory that will be recalled in a simplified form suited to our purposes.

2. The example

For a given $\nu \geq 1$, consider ν positive real numbers $\mu_1, \dots, \mu_\nu$ which are **rationally independent**. Let $\gamma \in \mathbb{R}$, and for a given integer $d \geq 1$, consider a real homogeneous polynomial

$$Q(x) = \sum_{|\alpha|=d} c_\alpha x^{2\alpha}, \quad x \in \mathbb{R}^\nu, \tag{2.1}$$

with nonnegative coefficients c_α satisfying

$$\sum_{|\alpha|=d} c_\alpha > 0. \tag{2.2}$$

Put $(D = -i\partial)$

$$X_j = D_{x_j} - i\mu_j x_j D_y, \quad j = 1, \dots, \nu, \tag{2.3}$$

and consider in $\mathbb{R}^{\nu+1} = \mathbb{R}^\nu_x \times \mathbb{R}_y$ the differential operator

$$A := \sum_{j=1}^{\nu} X_j^* X_j + \sum_{j=1}^{\nu} X_j \Big(Q(x) X_j^* \Big) + (\gamma + |\mu|) D_y, \tag{2.4}$$

where X_j^* is the adjoint of X_j and $|\mu| := \sum_{j=1}^{\nu} \mu_j$. Note that $A = A^*$ and that we may rewrite A as

$$\begin{aligned} A = (1 + Q(x)) \sum_{j=1}^{\nu} \big(D_{x_j}^2 + \mu_j^2 x_j^2 D_y^2 \big) \\ + (\gamma + |\mu| Q(x)) D_y + \frac{1}{i} \sum_{j=1}^{\nu} \frac{\partial Q}{\partial x_j}(x) \big(D_{x_j} + i\mu_j x_j D_y \big). \end{aligned} \tag{2.5}$$

The principal symbol of A vanishes exactly to second order on the symplectic manifold $\Sigma = \{(x, y; \xi, \eta);\ x = \xi = 0,\ \eta \neq 0\} \subset T^*\mathbb{R}^{\nu+1} \setminus 0$.

Of course, we are interested in the hypoellipticity of A at points of Σ.

From the classical result of Boutet de Monvel, Grigis and Helffer [1], it follows that A *is hypoelliptic at every point of* Σ *with loss of* 1 *derivative iff*

$$\gamma \notin \{\pm(2\langle \ell, \mu \rangle + |\mu|);\ \ell \in \mathbb{Z}^\nu_+\} \quad (\langle \ell, \mu \rangle := \sum_{j=1}^{\nu} \ell_j \mu_j). \tag{2.6}$$

We suppose that for some $\ell \in \mathbb{Z}^\nu_+$ (necessarily unique by virtue of the rational independence of the μ_j's)

$$\gamma = -2\langle \ell, \mu \rangle - |\mu|. \tag{2.7}$$

We will prove the following result.

Theorem 2.1. *Under condition* (2.7), *the operator* A *defined in* (2.4) *is hypoelliptic at any given point* $(x = 0, y) \in \mathbb{R}^{\nu+1}$ *with loss of* $d + 1$ *derivatives, and it cannot be hypoelliptic with a loss of fewer derivatives.*

The same result holds in case $\gamma = 2\langle \ell, \mu \rangle + |\mu|$.

As mentioned in the introduction, Theorem 2.1 is actually an *elementary* consequence of the general results of [6]. To make this paper as much self-contained as possible, we recall in the next section a simplified version of our machinery, postponing the proof of Theorem 2.1 to Section 4 below.

3. The machinery

Suppose we are given in $\mathbb{R}^n = \mathbb{R}^\nu_x \times \mathbb{R}^{n-\nu}_y$ $(1 \le \nu < n)$ a (properly supported) classical ψdo $A \in \mathrm{OPS}^m$ with symbol $\sigma(A)(x,\xi,\eta)$ **independent of** y

$$\sigma(A)(x,\xi,\eta) \sim \sum_{j\ge 0} a_{m-j/2}(x,\xi,\eta),$$

such that for some even integer $k \ge 2$ we have

$$\begin{cases} |a_{m-j/2}(x,\xi,\eta)| \lesssim (|\xi|+|\eta|)^{m-j/2}\left(|x|+\dfrac{|\xi|}{|\eta|}\right)^{k-j}, \ 0\le j\le k, \\ |a_m(x,\xi,\eta)| \gtrsim (|\xi|+|\eta|)^m \left(|x|+\dfrac{|\xi|}{|\eta|}\right)^k. \end{cases} \tag{3.1}$$

We study the hypoellipticity of A at points of the symplectic characteristic set $\Sigma = \{(x=0,y;\xi=0,\eta);\ y\in\mathbb{R}^{n-\nu}, \eta\in\mathbb{R}^{n-\nu}\setminus\{0\}\}$.

It is known after Sjöstrand [7] that A cannot be hypoelliptic at $\rho = (x = 0, y; \xi = 0, \eta) \in \Sigma$ with a loss of derivatives smaller than $k/2$. The hypoellipticity with loss of $k/2$ derivatives is characterized by Boutet-Grigis-Helffer's theorem [1] in terms of spectral properties of the **localized operator** $A^{(k)}_\eta$ **of** A **at** ρ defined as

$$A^{(k)}_\eta := \sum_{|\alpha|+|\beta|+j=k} \frac{1}{\alpha!\beta!}(\partial^\alpha_x\partial^\beta_\xi a_{m-j/2})(x=0,\xi=0,\eta)x^\alpha D^\beta_x. \tag{3.2}$$

Precisely, one has the following theorem.

Theorem 3.1. *A is hypoelliptic at $\rho=(x=0,y;\xi=0,\eta)$ with loss of $k/2$ derivatives iff $A^{(k)}_\eta : \mathcal{S}(\mathbb{R}^\nu) \longrightarrow \mathcal{S}(\mathbb{R}^\nu)$ is injective.*

Note that in the present setting the hypoellipticity condition does not depend on $y \in \mathbb{R}^{n-\nu}$.

To fix ideas, suppose now that for some $\eta_0 \in \mathbb{R}^{n-\nu}\setminus\{0\}$, $A^{(k)}_{\eta_0}$ is **not** injective, and suppose that the following hypothesis (H) be satisfied:

Hypothesis (H)**:**

(i) $[A^{(k)}_\eta, (A^{(k)}_\eta)^*] = 0$ *for all* $\eta \ne 0$;

(ii) *There exist functions* $u(\eta;x) = u \in C^\infty(\mathbb{R}^{n-\nu}_\eta\setminus\{0\};\mathcal{S}(\mathbb{R}^\nu_x))$ *and* $\lambda(\eta)=\lambda \in C^\infty(\mathbb{R}^{n-\nu}_\eta\setminus\{0\};\mathbb{C})$ *satisfying*

$$u(t\eta;t^{-1/2}x) = t^{\nu/4}u(\eta;x),\ t>0, \quad \int_{\mathbb{R}^\nu}|u(\eta;x)|^2dx = 1, \quad \forall\eta\ne 0,$$

$$\lambda(t\eta) = t^{m-k/2}\lambda(\eta),\ t>0, \quad \lambda(\eta_0)=0;$$

(iii) *For some conic neighborhood* $V \subset \mathbb{R}^{n-\nu}_\eta \setminus \{0\}$ *of* η_0 *we have*

$$(A^{(k)}_\eta u)(\eta; x) = \lambda(\eta) u(\eta; x), \quad \forall \eta \in V. \tag{3.3}$$

Note that (i) is satisfied whenever $A = A^*$ and that if (3.3) holds we also have

$$((A^{(k)}_\eta)^* u)(\eta; x) = \overline{\lambda(\eta)} u(\eta; x), \ \forall \eta \in V.$$

We next need the Hermite and co-Hermite operators.

Take a smooth symbol $\phi(\eta; x)$ having the asymptotic expansion

$$\phi(\eta; x) \sim \sum_{j \geq 0} \phi_{-j/2}(\eta; x),$$

such that

$$\begin{cases} \phi_{-j/2} \in C^\infty(\mathbb{R}^{n-\nu}_\eta \setminus \{0\}; \mathcal{S}(\mathbb{R}^\nu_x)), \\ \phi_{-j/2}(t\eta; t^{-1/2}x) = t^{\nu/4 - j/2} \phi_{-j/2}(\eta; x), \ t > 0, \\ \phi_0(\eta; x) = u(\eta; x). \end{cases} \tag{3.4}$$

The **Hermite operator** H^-_ϕ is defined by

$$\begin{cases} H^-_\phi : C^\infty_0(\mathbb{R}^{n-\nu}_y) \longrightarrow C^\infty(\mathbb{R}^n_{(x,y)}), \\ (H^-_\phi f)(x, y) = (2\pi)^{-(n-\nu)} \displaystyle\int e^{i\langle y, \eta\rangle} \phi(\eta; x) \hat{f}(\eta) \, d\eta, \end{cases} \tag{3.5}$$

and the **co-Hermite operator** H^+_ϕ by

$$\begin{cases} H^+_\phi : C^\infty_0(\mathbb{R}^n_{(x,y)}) \longrightarrow C^\infty(\mathbb{R}^{n-\nu}_y), \\ (H^+_\phi g)(y) = (2\pi)^{-(n-\nu)} \displaystyle\iint e^{i\langle y, \eta\rangle} \overline{\phi(\eta; x)} \hat{g}(x, \eta) \, d\eta dx. \end{cases} \tag{3.6}$$

Fix ϕ as above and consider the system

$$\mathcal{A}_\phi = \begin{bmatrix} A & H^-_\phi \\ H^+_\phi & 0 \end{bmatrix} : \begin{matrix} C^\infty_0(\mathbb{R}^n) \\ \times \\ C^\infty_0(\mathbb{R}^{n-\nu}) \end{matrix} \longrightarrow \begin{matrix} C^\infty(\mathbb{R}^n) \\ \times \\ C^\infty(\mathbb{R}^{n-\nu}) \end{matrix}. \tag{3.7}$$

We have the following theorem (see [6], Thm. 4.6).

Theorem 3.2. *Given* $\mathcal{A}_\phi$ *as above there exist:*

(i) *a symbol* $\psi(\eta; x) \sim \sum_{j \geq 0} \psi_{-j/2}(\eta; x)$ *satisfying* (3.4)*;*

(ii) *a* ψdo $E \in \mathrm{OPS}^{-m+k/2}_{1/2,1/2}(\mathbb{R}^n)$ *which is microlocal (i.e.,* $\mathrm{WF}(Ev) \subset \mathrm{WF}(v)$ *for all* $v \in \mathcal{E}'(\mathbb{R}^n)$*);*

(iii) *a classical* ψdo $\Lambda \in \mathrm{OPS}^{m-k/2}(\mathbb{R}^{n-\nu})$ *with symbol* $\sigma(\Lambda)(\eta)$ *independent of* y *and principal symbol* $\Lambda_{m-k/2}(\eta) = \lambda(\eta)$, *such that the system*

$$\mathcal{E}_\psi = \begin{bmatrix} E & H_\psi^- \\ H_\psi^+ & -\Lambda \end{bmatrix} \tag{3.8}$$

is a microlocal two-sided parametrix of $\mathcal{A}_\phi$ on an open conic set Γ_ε, $\varepsilon > 0$, of the form

$$\Gamma_\varepsilon = \left\{ (x, y; \xi, \eta) \in T^*\mathbb{R}^n;\ \eta \neq 0,\ |x| + \frac{|\xi|}{|\eta|} < \varepsilon,\ \left| \frac{\eta}{|\eta|} - \frac{\eta_0}{|\eta_0|} \right| < \varepsilon \right\}.$$

As a consequence of Theorem 3.2 we have that *the operator A is hypoelliptic at $(0, y, 0, \eta_0)$ with loss of $\frac{k}{2} + r$ $(r > 0)$ derivatives iff the operator Λ is hypoelliptic at (y, η_0) with loss of r derivatives* (see [6], Thm. 5.1).

In particular, when ϕ is chosen such that $\phi_{-j/2} = 0$, $j \geq 1$, we have the following description of the total symbol of Λ for $|\eta/|\eta| - \eta_0/|\eta_0|| < \varepsilon$:

$$\begin{cases} \Lambda_{m-k/2}(\eta) = \lambda(\eta), \\ \Lambda_{m-k/2-j/2}(\eta) = \displaystyle\sum_{\substack{p+q=j \\ 0 \leq q < j}} \left(A_\eta^{(k+p)} \psi_{-q/2}(\eta; \cdot), \phi_0(\eta; \cdot) \right)_{L^2(\mathbb{R}^\nu)},\ j \geq 1, \end{cases} \tag{3.9}$$

with

$$\begin{cases} \psi_0 = \phi_0 = u, \\ \psi_{-q/2}(\eta; x) = (A_\eta^{(k)})^{-1} \Big[\Lambda_{m-k/2-q/2}(\eta)\phi_0(\eta; \cdot) + \\ \qquad\qquad - \sum_{\substack{p' + q' = q \\ 0 \leq q' < q}} A_\eta^{(k+p')} \psi_{-q'/2}(\eta; \cdot) \Big],\ q \geq 1, \end{cases} \tag{3.10}$$

where

$$A_\eta^{(k+p)} := \sum_{|\alpha|+|\beta|+j=k+p} \frac{1}{\alpha!\beta!} (\partial_x^\alpha \partial_\xi^\beta a_{m-j/2})(x = 0, \xi = 0, \eta) x^\alpha D_x^\beta,\ p \geq 0, \tag{3.11}$$

are (for $p \geq 1$) the *higher-order localized operators*, and $(A_\eta^{(k)})^{-1}$ is the inverse of the restriction of $A_\eta^{(k)}$ to $[\mathrm{Ker}(A_\eta^{(k)} - \lambda(\eta))]^\perp$.

4. Proof of Theorem 2.1

We now have $m = k = 2$, $n = \nu + 1$. From (2.7) and Theorem 3.1 we already have that A is hypoelliptic at the points $(x = 0, y, \xi = 0, \eta < 0)$ with loss of 1 derivative. On the other hand, $A_\eta^{(2)}$ is **not** injective for $\eta > 0$. Hypothesis (H) is fulfilled on $V = \{\eta > 0\}$ with

$$u(\eta; x) = h_\ell(\eta; x) := \prod_{j=1}^{\nu} (\mu_j |\eta|)^{1/4} h_{\ell_j}(\sqrt{\mu_j |\eta|}\, x_j) \tag{4.1}$$

(where

$$h_r(t) = \frac{1}{\pi^{1/4}\sqrt{2^r r!}}\Big(\frac{d}{dt} - t\Big)^r (e^{-t^2/2}),\ r \in \mathbb{Z}_+,\ t \in \mathbb{R},$$

are the usual Hermite functions) and

$$\lambda(\eta) = 0, \quad \text{for} \quad \eta > 0. \tag{4.2}$$

An elementary computation shows that

$$A_\eta^{(2+p)} = \begin{cases} 0, \quad 1 \le p \ne 2d, \\ Q(x)\displaystyle\sum_{j=1}^{\nu}(D_{x_j}^2 + \mu_j^2 x_j^2 \eta^2) + |\mu| Q(x) \\ \qquad\qquad\qquad + \dfrac{1}{i}\displaystyle\sum_{j=1}^{\nu}\frac{\partial Q}{\partial x_j}(x)(D_{x_j} + i\mu_j x_j \eta),\ p = 2d. \end{cases} \tag{4.3}$$

From (3.9) we get

$$\Lambda_{1-j/2}(\eta) = 0,\ \eta > 0,\ 0 \le j \le 2d-1, \tag{4.4}$$

so that from (3.10) the $\psi_{-q/2} = 0$ for $\eta > 0$, $1 \le q \le 2d-1$, thereby yielding that for $\eta > 0$

$$\begin{aligned} \Lambda_{1-d}(\eta) &= \Big(A_\eta^{(2+2d)} u(\eta;\cdot), u(\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \\ &= 2\langle \ell, \mu\rangle |\eta| \Big(Q(\cdot) h_\ell(\eta;\cdot), h_\ell(\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)} \\ &\quad - \sum_{j=1}^{\nu} \sqrt{2(\ell_j+1)\mu_j|\eta|}\,\Big(\frac{\partial Q}{\partial x_j}(\cdot) h_{\ell+e_j}(\eta;\cdot), h_\ell(\eta;\cdot)\Big)_{L^2(\mathbb{R}^\nu)}. \end{aligned} \tag{4.5}$$

By the change of variables $\sqrt{\mu_j|\eta|}\,x_j = t_j$, $1 \le j \le \nu$, and upon defining

$$\tilde{Q}(t) := \sum_{|\alpha|=d} \frac{c_\alpha}{\mu^{\alpha/2}} t^{2\alpha} =: \sum_{|\alpha|=d} \tilde{c}_\alpha t^{2\alpha},\ t \in \mathbb{R}^\nu, \tag{4.6}$$

we have for $\eta > 0$

$$\begin{aligned} \Lambda_{1-d}(\eta) = |\eta|^{1-d}\Bigg(& 2\langle \ell, \mu\rangle \int_{\mathbb{R}^\nu} \tilde{Q}(t) h_\ell(t)^2 dt \\ & - \sum_{j=1}^{\nu} \mu_j \sqrt{2(\ell_j+1)} \int_{\mathbb{R}^\nu} \frac{\partial \tilde{Q}}{\partial t_j}(t) h_{\ell+e_j}(t) h_\ell(t) dt \Bigg). \end{aligned} \tag{4.7}$$

Now, since

$$t h_r(t) = -\frac{1}{2}\Big(\sqrt{2r}\,h_{r-1}(t) + \sqrt{2(r+1)}\,h_{r+1}(t)\Big),$$

we have, for $j = 1, 2, \ldots, \nu$,

$$\begin{aligned}
&\int_{\mathbb{R}^\nu} \frac{\partial \tilde{Q}}{\partial t_j}(t) h_{\ell+e_j}(t) h_\ell(t) dt \qquad (4.8)\\
&= \sum_{\substack{|\alpha| = d \\ \alpha_j \geq 1}} \alpha_j \tilde{c}_\alpha \left(\prod_{r \neq j} \int_{\mathbb{R}} t_r^{2\alpha_r} h_{\ell_r}(t_r)^2 dt_r \right) \int_{\mathbb{R}} t_j^{2\alpha_j - 1} h_{\ell_j+1}(t_j) h_{\ell_j}(t_j) dt_j \leq 0,
\end{aligned}$$

with strict inequality for some j, by virtue of (2.2). From (4.7) and (4.8) we hence conclude that

$$\Lambda_{1-d}(\eta) > 0, \quad \text{for} \quad \eta > 0. \tag{4.9}$$

Thus, from (4.4) and (4.9) we have that Λ is hypoelliptic at every point $(y, \eta > 0)$ with loss of d derivatives and no fewer, whence the proof of Theorem 2.1. $\square$

Remark 4.1.

1. It is worth observing that if we consider $A + \delta$, δ a non-zero complex number, then $A + \delta$ is hypoelliptic with loss of 2 derivatives, regardless the degree of the polynomial Q. Also, the results of Grigis and Rothschild [4] apply in this case, yielding the analytic hypoellipticity of $A + \delta$.
2. We believe that under the hypotheses of Theorem 2.1, A is also analytic hypoelliptic. This conjecture is supported by the results of Grigis and Rothschild [4], and by the very recent result on Kohn's example by Derridj and Tartakoff [3], which appears as an Appendix to the paper of Kohn [5], and a still more recent and more general paper by Tartakoff [8], which proves analyticity when the vector fields are only of finite type.
3. In case the μ_j's are **rationally dependent** and **there are** $N \geq 2$ multi-indices $\ell^{(j)} \in \mathbb{Z}_+$ for which $\gamma = -\langle \ell^{(j)}, \mu \rangle - |\mu|$, $j = 1, \ldots, N$, the operator Λ is actually an $N \times N$ system of first order ψdo's. Again, $\Lambda_{1-j/2}(\eta) = 0$ for $\eta > 0$, $0 \leq j \leq 2d - 1$, but $\Lambda_{1-d}(\eta)$ $(\eta > 0)$ is an $N \times N$ matrix whose invertibility is much more painful to establish.
4. It is conceivable that, in view of the result by Christ [2], the operator $A + D_s^2$ in $\mathbb{R}_x^\nu \times \mathbb{R}_y \times \mathbb{R}_s$ is no longer hypoelliptic.

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On the Nonstationary Two-dimensional Navier-Stokes Problem in Domains with Strip-like Outlets to Infinity

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1. Introduction

In this paper we study the following initial-boundary value problem for nonstationary Navier–Stokes system

$$\left\{\begin{aligned} \mathbf{u}_t-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f},\\ \mathrm{div}\mathbf{u}(x,t)=0,\\ \mathbf{u}(x,t)\big|_{\partial\Omega}=0,\quad \mathbf{u}(x,0)=\mathbf{u_0}(x),\\ \int_{\sigma_j}\mathbf{u}(x,t)\cdot\mathbf{n}(x)\,dS=F_j(t),\ j=1,\dots,J,\\ \textstyle\sum_{j=1}^J F_j(t)=0\quad\forall t\in[0,T]\end{aligned}\right. \tag{1.1}$$

in a domain $\Omega\subset\mathbb{R}^2$ with J strip-like outlets to infinity, i.e., for sufficiently large $|x|$ the domain Ω splits into J disconnected components Ω_j (outlets to infinity) that in certain coordinate systems $x^{(j)}$ have the form $\Omega_j=\{x^{(j)}\in\mathbb{R}^2:\ x_1^{(j)}\in\sigma_j,\ 0<x_2^{(j)}<\infty\ \}$ where $\sigma_j=(0,\ h_j)$. The condition (1.1_4) prescribes fluxes of the velocity vector $\mathbf{u}(x,t)$ over cross-sections σ_j of outlets to infinity Ω_j and the condition (1.1_5) means that the total flux is equal to zero for all $t\in[0,\ T]$.

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We assume that the initial velocity $\mathbf{u_0}$ and the external force $\mathbf{f}$ admit the representations

$$\begin{aligned} \mathbf{u}_0(x) &= \sum_{j=0}^{J} \zeta(x_2^{(j)})\big(0,\, u_{02}^{(j)}(x_1^{(j)})\big) + \widehat{\mathbf{u}}_0(x), \\ \mathbf{f}(x,t) &= \sum_{j=0}^{J} \zeta(x_2^{(j)})\big(0,\, f_2^{(j)}(x_1^{(j)},t)\big) + \widehat{\mathbf{f}}(x,t), \end{aligned} \tag{1.2}$$

where $\zeta(\tau)$ is a smooth cut-off function with $\zeta(\tau)=0$ for $\tau\le 1$ and $\zeta(\tau)=1$ for $\tau\ge 2$, $u_{02}^{(j)}\in W_2^3(\sigma_j)$, $f_2,\ f_{2t}\in L_2(\Sigma_j^T)$, $\Sigma_j^T=\sigma_j\times(0,T)$, and $\widehat{\mathbf{f}}$, $\widehat{\mathbf{f}}_t$ and $\widehat{\mathbf{u}}_0$ belong to certain weighted spaces of vanishing at infinity functions. Moreover, we suppose that there hold the compatibility conditions

$$\begin{aligned} \operatorname{div}\mathbf{u}_0(x) = 0, \qquad \mathbf{u}_0(x)\big|_{\partial\Omega} &= 0, \\ \big(\nu u_{02}^{(j)\prime\prime}(x_1^{(j)}) + f_2^{(j)}(x_1^{(j)},\,0)\big)\big|_{\partial\sigma_j} &= 0, \\ F_j(0) = \int_0^{h_j} u_{02}^{(j)}(x_1^{(j)})\,dx_1^{(j)}&, \\ F_j'(0) = \int_0^{h_j} \big(\nu u_{02}^{(j)\prime\prime}(x_1^{(j)}) + f_2^{(j)}(x_1^{(j)},\,0)\big)\,dx_1^{(j)},\ j&=1,\ldots,J. \end{aligned} \tag{1.3}$$

Under these assumptions we prove that problem (1.1) has a solution $\mathbf{u}(x,t)$ which tends in each outlet to infinity Ω_j to a nonstationary Poiseuille flow corresponding to this outlet.

Nonstationary Poiseuille flow in a strip $\Pi=\{x\in\mathbb{R}^2\ :\ x_1\in\sigma=(0,\ h), -\infty< x_2<\infty\}$ is an exact solution of nonstationary Navier–Stokes system having the form

$$\mathbf{U}(x,t)=(0,\,U(x_1,t)), \qquad P(x,t)=-q(t)x_2+p_0(t),$$

where the pair $\big(U(x_1,t),\,q(t)\big)$ is a solution of the following inverse problem

$$\begin{cases} U_t(x_1,t)-\nu\frac{\partial^2 U(x_1,t)}{\partial x_1^2}=q(t)+f_2(x_1,t), \\ U(x_1,t)\big|_{\partial\sigma}=0, \quad U(x_1,0)=u_{02}(x_1), \\ \int_\sigma U(x_1,t)\,dx_1=F(t) \end{cases} \tag{1.4}$$

and $p_0(t)$ is an arbitrary function. In (1.4) $U(x_1,t),\ q(t)$ are unknown functions and $F(t)$ is a prescribed flux of $\mathbf{U}(x,t)$ over the cross-section σ. Under compatibility conditions (1.3) the unique solvability of problem (1.4) in Hölder spaces is proved in [4], in Sobolev spaces in [9]. In [10] the behavior of this solution as $t\to\infty$ is investigated. The existence of time-periodic Poiseuille flow in Sobolev spaces is proved in [2, 3]. The mentioned results are obtained also for the three-dimensional case.

Results of the paper are related to the famous Leray problem for Poiseuille flow. Let us assume that the domain $\Omega\subset\mathbb{R}^n$, $n=2,3$, consists of two semi-infinite cylinders Π_j, $j=1,2$, with cross sections σ_j connected by the bounded pipe Ω_0. The Leray problem consists of finding the solution $(\mathbf{u},p)$ of the steady Navier–Stokes problem that tends as $|x|\to\infty, x\in\Pi_j$, to steady Poiseuille flows $(\mathbf{U}_F^{(j)}(x^{(j)\prime}), P_F^{(j)}(x_n^{(j)}))$ corresponding to Π_j, $j=1,2$. Here $(x^{(j)\prime},\,x_n^{(j)})=(x_1^{(j)},\ldots,x_{n-1}^{(j)},x_n^{(j)})$ are local coordinates in $\mathbb{R}^n$ with $x_n^{(j)}$ directed along the axis of the cylinder Π_j and σ is an arbitrary cross-section of Ω. The first result concerning Leray's problem was obtained by Amick [1] who proved that this problem has a unique solution, if the flux $|F|$ is sufficiently small (in comparison with the viscosity ν). The most general results concerning the stationary Leray problem were obtained by Ladyzhenskaya and Solonnikov [6]. They have considered the steady Navier–Stokes problem in a domain Ω with J cylindrical outlets to infinity Π_j, $j = 1, \dots, J$, and have proved the existence of a solution with an infinite Dirichlet integral having prescribed fluxes F_j. This result is obtained without any restrictions on values of fluxes F_j, assuming only the necessary condition that the total flux is equal to zero, i.e., $\sum_{j=1}^{J} F_j = 0$. Moreover, it is proved in [6] that for sufficiently small $|\mathbf{F}| = \sqrt{\sum_{j=1}^{N} |F_j|^2}$ the obtained solution exponentially tends in each pipe to the corresponding Poiseuille flow.

In this paper we solve the nonstationary Leray problem in a two-dimensional domain with strip-like outlets to infinity globally in time for arbitrary fluxes, initial data and external forces satisfying only the necessary compatibility conditions (1.3). In particular, from the obtained results it follows that, if $\widehat{\mathbf{f}}(x,t) = 0$, $\widehat{\mathbf{u}}_0(x) = 0$ (or $\widehat{\mathbf{f}}(x,t)$ and $\widehat{\mathbf{u}}_0(x)$ vanish exponentially), then the solution $\mathbf{u}(x,t)$ tends in each Ω_j to the corresponding Poiseuille flow exponentially as $|x| \to \infty$. The analogous results (also for the three-dimensional case) for the linearized nonstationary Navier-Stokes system were obtained in [11].

The properties of solutions to the nonstationary Navier–Stokes system in domains with noncompact boundaries are studied not nearly enough. It is known (see [7, 8, 12, 13]) that in domains with outlets to infinity there exist solutions with prescribed fluxes $F_j(t)$ and, dependent on the geometry of the outlets, solutions have finite or infinite energy integral. In particular, if outlets are cylindrical, the energy integral is infinite. Note that the solvability of both two- and three-dimensional nonlinear nonstationary Navier–Stokes problems is proved in [7, 8, 12, 13] either for small data or for small time intervals. The global solvability of the two-dimensional nonstationary Navier–Stokes problem in domains with cylindrical outlets to infinity is mentioned in [13] as unsolved problem.

2. Notations and auxiliary results

2.1. Definitions of function spaces

Let V be a Banach space. The norm of an element u in V is denoted by $\|u;\ V\|$. Vector-valued functions are denoted by bold letters and spaces of scalar and vector-valued functions are not distinguished in notations. A vector-valued function $\mathbf{u} = (u_1, \dots, u_n)$ belongs to the space V, if $u_i \in V, i = 1, \dots, n$, and $\|\mathbf{u};\ V\| = \sum_{i=1}^{n} \|u_i;\ V\|$.

Let G be an arbitrary domain in $\mathbb{R}^n, n \geq 1$, with the boundary ∂G. As usual, denote by $C^\infty(G)$ a set of all infinitely differentiable functions in G and by $C_0^\infty(G)$ a subset of functions from $C^\infty(G)$ with compact supports in G. $W_p^l(G)$, $l \geq 0$, $p > 1$, is a usual Sobolev space, $L_p(G) = W_p^0(G)$ and $\mathring{W}_p^1(G)$ is the closure of $C_0^\infty(G)$ in the norm $\|\cdot;\ W_p^1(G)\|$.

Let $\Omega \subset \mathbb{R}^2$ be a domain with J cylindrical outlets to infinity, i.e., outside the sphere $|x| = r_0$ the domain Ω splits into J connected components Ω_j (outlets to infinity) that in some coordinate systems $x^{(j)}$ are given by the relations

$$\Omega_j = \big\{x^{(j)} = (x_1^{(j)}, x_2^{(j)}) \in \mathbb{R}^2,\ x_1^{(j)} \in \sigma_j,\ \ x_2^{(j)} > 0\big\},$$

where $\sigma_j = (0,\ h_j)$ are intervals. We introduce the following notations:

$$\Omega_{jk} = \{x \in \Omega_j :\ x_2^{(j)} < k\}, \qquad \omega_{jk} = \Omega_{jk+1} \setminus \Omega_{jk}, \quad j = 1, \dots, J,$$
$$\Omega_{(k)} = \Omega_{(0)} \bigcup \Big(\bigcup\nolimits_{j=1}^{J} \Omega_{jk} \Big), \qquad \Omega_{(0)} = \Omega \cap \{x :\ |x| < r_0\}, \quad Q^T = \Omega \times (0, T),$$

where $k \geq 0$ is an integer.

Denote $\boldsymbol{\beta} = (\beta_1, \dots, \beta_J)$ and let $E_{\beta_j}(x) = E_{\beta_j}(x_2^{(j)})$ be smooth weight functions in Ω_j satisfying the conditions

$$\begin{aligned} E_{\beta_j}(x) > 0, \quad a_1 \leq E_{-\beta_j}(x) E_{\beta_j}(x) \leq a_2 \quad \forall x \in \Omega_j, \quad E_{\beta_j}(0) = 1, \\ b_1 E_{\beta_j}(k) \leq E_{\beta_j}(x) \leq b_2 E_{\beta_j}(k) \qquad \forall x \in \omega_{jk}, \\ |\nabla E_{\beta_j}(x)| \leq b_3 \gamma_* E_{\beta_j}(x) \qquad \forall x \in \Omega_j, \\ \lim\nolimits_{x_2^{(j)} \to \infty} E_{\beta_j}(x) = \infty, \qquad \text{if} \quad \beta_j > 0, \end{aligned} \tag{2.1}$$

where the constants a_1, a_2, b_1, b_2 are independent of k, b_3 is independent of β_j and $\gamma_* > 0$ is sufficiently small. Simple examples of such weight functions are

$$E_{\beta_j}(x) = \big(1 + \delta^2 |x_2^{(j)}|^2\big)^{\beta_j} \quad \text{and} \quad E_{\beta_j}(x) = \exp\big(2\beta_j x_2^{(j)}\big).$$

The conditions $(2.1_1), (2.1_2)$ and (2.1_4) for these functions are obvious. The condition (2.1_3) for the first weight function is valid, if $|\beta_j|\delta \leq \gamma_*$, and for the second one, if $|\beta_j| \leq \gamma_*$. Let us put

$$E_{\boldsymbol{\beta}}(x) = \begin{cases} 1, & x \in \Omega_{(0)}, \\ E_{\beta_j}(x), & x \in \Omega_j, \ \ j = 1, \dots, J, \end{cases} \tag{2.2}$$

and define in Ω weighted function spaces. Denote by $\mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega)$, $l \geq 0$, the space of functions obtained as a closure of $C_0^\infty(\Omega)$ in the norm

$$\|u;\ \mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega)\| = \bigg(\sum_{|\alpha|=0}^{l} \int_\Omega E_{\boldsymbol{\beta}}(x) |D^\alpha u(x)|^2 \, dx \bigg)^{1/2}.$$

A weight index $\beta_j > 0$ shows the decay rate as $|x| \to \infty,\ x \in \Omega_j$, of elements $u \in \mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega)$. For example, if $E_{\beta_j}(x) = \exp\big(2\beta_j x_2^{(j)}\big),\ \beta_j > 0$, then elements from $\mathcal{W}^2_{2,\boldsymbol{\beta}}(\Omega_j)$ vanish exponentially as $x \to \infty,\ x \in \Omega_j$. Obviously,

$$\mathcal{W}^l_{2,\boldsymbol{\beta}}(\Omega) \subset W^l_2(\Omega) \subset \mathcal{W}^l_{2,-\boldsymbol{\beta}}(\Omega) \quad \text{for} \quad \beta_j \geq 0,\ j = 1, \cdots, J.$$

We will need also a "step" weight function

$$E^{(k)}_{\boldsymbol{\beta}}(x) = \begin{cases} 1, & x \in \Omega_{(0)}, \\ E_{\beta_j}(x_2^{(j)}), & x \in \Omega_{jk}, \ \ j = 1, \dots, J, \\ E_{\beta_j}(k), & x \in \Omega_j \setminus \Omega_{jk}, \ \ j = 1, \dots, J. \end{cases} \tag{2.3}$$

It is easy to see that

$$|\nabla E^{(k)}_{\boldsymbol\beta}(x)| \le b_3\gamma_* E^{(k)}_{\boldsymbol\beta}(x), \tag{2.4}$$

Moreover, by the definition $\operatorname{supp} \nabla E^{(k)}_{\boldsymbol\beta} \subset \overline{\bigcup_{j=1}^J \Omega_{jk}}$.

2.2. Divergence equation

Below we will need results concerning the solvability of the divergence equation. There holds the following

Lemma 2.1. *Let* $\mathbf{u}(\,\cdot\,,t) \in \mathring{W}^1_2(\Omega)$, $\mathbf{u}_t(\,\cdot\,,t) \in L_2(\Omega)$, $\operatorname{div}\mathbf{u}(x,t) = 0 \ \ \forall t \in [0,T]$,

$$\int_{\sigma_j} \mathbf{u}(x,t)\cdot\mathbf{n}(x)\,ds = 0, \qquad j = 1,\dots,J. \tag{2.5}$$

Then there exists a vector-function $\mathbf{W}^{(k)}(\,\cdot\,,t) \in \mathring{W}^1_2(\Omega)$ *with* $\mathbf{W}^{(k)}_t(\,\cdot\,,t) \in \mathring{W}^1_2(\Omega)$ *such that* $\operatorname{supp}\mathbf{W}^{(k)} \subset \overline{\bigcup_{j=1}^J \Omega_{jk}}$ *and*

$$\operatorname{div}\mathbf{W}^{(k)}(x,t) = -\operatorname{div}\big(E^{(k)}_{\boldsymbol\beta}(x)\mathbf{u}(x,t)\big), \qquad x \in \Omega. \tag{2.6}$$

Moreover, there hold the estimates

$$\begin{aligned}\int_\Omega E^{(k)}_{-\boldsymbol\beta}(x)|\nabla\mathbf{W}^{(k)}(x,t)|^2\,dx &\le c\gamma_*^2\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\mathbf{u}(x,t)|^2\,dx\\ &\le c\gamma_*^2\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\nabla\mathbf{u}(x,t)|^2\,dx, \\ \int_\Omega E^{(k)}_{-\boldsymbol\beta}(x)|\nabla\mathbf{W}^{(k)}_t(x,t)|^2\,dx &\le c\gamma_*^2\int_\Omega E^{(k)}_{\boldsymbol\beta}(x)|\mathbf{u}_t(x,t)|^2\,dx.\end{aligned} \tag{2.7}$$

The constants in inequalities (2.7) *are independent of* k.

We need also the following result concerning the existence of "regular" solutions of the divergence equation.

Lemma 2.2. *Let* $\partial\Omega \in C^4$ *and let* $g(x,t)$ *have a finite norm*

$$\sup_{t\in[0,T]}\Big(\|g(\,\cdot,t);\ W^2_2(\Omega)\|^2 + \|g_t(\cdot\,,t);\ W^1_2(\Omega)\|^2\Big) + \int_0^T \|g(\,\cdot,t);\ W^2_2(\Omega)\|^2\,dt$$
$$+\int_0^T \|g_t(\,\cdot,t);\ W^1_2(\Omega)\|^2\,dt + \int_0^T \|g_{tt}(\,\cdot,t);\ L_2(\Omega)\|^2\,dt := |||g|||^2 < \infty.$$

Suppose that

$$\operatorname{supp}_x g(\cdot,t) \subset \overline{\Omega}_{(2)}, \qquad \int_{\Omega_{(2)}} g(x,t)dx = 0 \quad \forall t\in[0,\,T]. \tag{2.8}$$

Then there exists a vector-field $\mathbf{W}$ *such that*

$$\begin{aligned}\operatorname{div}\mathbf{W}(x,t) = g(x,t), \quad x\in\Omega; \qquad \mathbf{W}(x,t)\big|_{\partial\Omega} = 0,\\ \operatorname{supp}_x \mathbf{W}(\cdot,t) \subset \overline{\Omega}_{(3)} \quad \forall t\in[0,\,T]\end{aligned} \tag{2.9}$$

and there holds the estimate

$$\sup_{t\in[0,T]}\Big(\|\mathbf{W}(\,\cdot,t);\ W_2^3(\Omega)\|^2+\|\mathbf{W}_t(\cdot\,,t);\ W_2^2(\Omega)\|^2\Big)$$
$$+\int_0^T\Big(\|\mathbf{W}(\,\cdot,t);\ W_2^3(\Omega)\|^2+\|\mathbf{W}_t(\,\cdot,t);\ W_2^2(\Omega)\|^2$$
$$+\|\mathbf{W}_{tt}(\,\cdot,t);\ W_2^1(\Omega)\|^2\Big)\,dt\le c\,|||g|||^2.$$

2.3. Nonstationary Poiseuille flow in an infinite strip

Let $\Pi=\big\{x\in\mathbb{R}^2\ :\ x_1\in\sigma=(0,\,h),\ x_2\in\mathbb{R}\big\}$ be an infinite strip in $\mathbb{R}^2$. Consider in $\Pi\times(0,\,T)$ the nonstationary Navier-Stokes problem

$$\begin{cases}\mathbf{u}_t-\nu\Delta\mathbf{u}+(\mathbf{u}\cdot\nabla)\mathbf{u}+\nabla p=\mathbf{f},\\ \mathrm{div}\mathbf{u}(x,t)=0,\\ \mathbf{u}(x,t)\big|_{\partial\Pi}=0,\quad \mathbf{u}(x,0)=\mathbf{u_0}(x),\\ \int_0^h u_2(x_1,x_2,t)\,dx_1=F(t).\end{cases}\tag{2.10}$$

Assume that data do not depend on x_2 and have the form

$$\mathbf{f}(x,t)=(0,\ f_2(x_1,t)),\qquad \mathbf{u}_0(x)=(0\ ,u_{02}(x_1)).$$

Moreover, suppose that there holds the necessary compatibility condition

$$\int_0^h u_{02}(x_1)\,dx_1=F(0).\tag{2.11}$$

The nonstationary Poiseuille solution has the form

$$\mathbf{U}(x,t)=(0,\,U_2(x_1,t)),\qquad P(x,t)=-q(t)x_2+p_0(t),\tag{2.12}$$

where $p_0(t)$ is an arbitrary function of t. The pair $(U_2(x_1,t),q(t))$ is the solution of the following inverse problem in $\Sigma^T=\sigma\times(0,\ T)$:

$$\begin{cases}U_{2t}(x_1,t)-\nu\frac{\partial^2U_2(x_1,t)}{\partial x_1^2}=q(t)+f_2(x_1,t),\\ U_2(x_1,t)\big|_{\partial\sigma}=0,\qquad U_2(x_1,0)=u_{02}(x_1),\\ \int_0^h U_2(x_1,t)\,dx_1=F(t).\end{cases}\tag{2.13}$$

Problem (2.14) has been studied in [9]. Let us formulate the obtained results.

Theorem 2.3. *Let $\partial\sigma\in C^2$, $u_{02}\in W_2^3(\sigma)$, $f_2\in L_2(\Sigma^T)$, $f_{2t}\in L_2(\Sigma^T)$, $F\in W_2^2(0,T)$, and let there hold the compatibility conditions*

$$\begin{gathered}\big(\nu u_{02}''(x_1)+f_2(x_1,0)\big)\big|_{\partial\sigma}=0,\\ F(0)=\int_0^h u_{02}(x_1)\,dx_1,\qquad F'(0)=\int_0^h\big(\nu u_{02}''(x_1)+f_2(x_1,0)\big)\,dx_1.\end{gathered}\tag{2.14}$$

Then for arbitrary $T \in (0, \infty]$ *there exists a unique solution* (U_2, q) *of problem* (2.14) *satisfying the estimate*

$$\begin{gathered}\sup_{t\in[0,T]}\Big(\|U_2(\cdot,t);\ W_2^2(\sigma)\|^2+\|U_{2t}(\cdot,t);\ W_2^1(\sigma)\|^2\Big)\\ +\int_0^T\Big(\|U_2(\cdot,t);\ W_2^2(\sigma)\|^2+\|U_{2t}(\cdot,t);\ W_2^1(\sigma)\|^2\\ +\|U_{2tt}(\cdot,t);\ L_2(\sigma)\|^2\Big)\,dt+\|q;\ W_2^1(0,T)\|^2\le c\Big(\|F;\ W_2^2(0,T)\|^2\\ +\|u_{02};\ W_2^3(\sigma)\|^2+\|f_2;\ L_2(\Sigma^T)\|^2+\|f_{2t};\ L_2(\Sigma^T)\|^2\Big).\end{gathered} \tag{2.15}$$

2.4. Construction of the flux carrier in the domain Ω

Let us assume that $F_j(t) \in W_2^2(0,T)$ and that the initial velocity $\mathbf{u_0}$ and the external force $\mathbf{f}$ are represented in the form (1.2) where $\widehat{\mathbf{u}}_0 \in \mathcal{W}^2_{2,\boldsymbol{\beta}}(\Omega)$, $\widehat{\mathbf{f}},\ \widehat{\mathbf{f}}_t \in \mathcal{L}_{2,\boldsymbol{\beta}}(Q^T)$. Moreover, let $F_j(t)$ and $\big(u_{02}^{(j)}(x_1^{(j)}), f_2^{(j)}(x_1^{(j)},t)\big)$, $j=1,\dots,J$, satisfy the compatibility conditions (2.15). Then in each strip $\Pi_j = \sigma_j \times (0,T)$ there exists a nonstationary Poiseuille solution $\mathbf{U}^{(j)}(x,t) = \big(0,\, U_2^{(j)}(x_1^{(j)},t)\big)$, $P^{(j)}(x,t) = -q^{(j)}(t)x_2^{(j)} + p_0^{(j)}(t)$, where the pair $\big(U_2^{(j)}(x_1^{(j)},t),\, q^{(j)}(t)\big)$ is the solution of the inverse problem (2.14) in $\Sigma_j^T = \sigma_j \times (0,T)$ with $F(t) = F_j(t)$, $u_{02} = u_{02}^{(j)}(x_1^{(j)})$ and $f_2 = f_2^{(j)}(x_1^{(j)},t)$, $j=1,\dots,J$. By Theorem 2.3 for $\big(U_2^{(j)}(x_1^{(j)},t),\, q^{(j)}(t)\big)$ there holds the estimate (2.16).

Let us define

$$\mathbf{U}(x,t) = \sum_{j=1}^J \zeta(x_2^{(j)})\mathbf{U}^{(j)}(x_1^{(j)},t), \quad P(x,t) = \sum_{j=1}^J \zeta(x_2^{(j)})P^{(j)}(x,t). \tag{2.16}$$

Then

$$\begin{gathered}G(x,t) = -\operatorname{div}\mathbf{U}(x,t) = -\sum_{j=1}^J \zeta'(x_2^{(j)})\,U_2^{(j)}(x_1^{(j)},t),\\ \operatorname{supp}_x G(x,t) \subset \overline{\Omega_{(2)}\setminus\Omega_{(1)}}.\end{gathered} \tag{2.17}$$

Moreover, from the condition $\sum_{j=1}^J F_j(t) = 0$ it follows that

$$\int_{\Omega_{(2)}} G(x,t)\,dx = 0 \qquad \forall t\in[0,T]. \tag{2.18}$$

Let $\partial\Omega \in C^4$. Then, due to Lemma 2.2, there exists a vector-field $\mathbf{W}(x,t)$ such that

$$\operatorname{div}\mathbf{W}(x,t) = G(x,t), \quad \mathbf{W}(x,t)\big|_{\partial\Omega} = 0, \quad \operatorname{supp}_x \mathbf{W}(\cdot,t) \subset \overline{\Omega}_{(3)}$$

and

$$\sup_{t\in[0,T]}\Big(\|\mathbf{W}(\cdot,t);\ W_2^3(\Omega_{(3)})\|^2+\|\mathbf{W}_t(\cdot,t);\ W_2^2(\Omega_{(3)})\|^2\Big)$$
$$+\int_0^T\Big(\|\mathbf{W}(\cdot,t);\ \mathbf{W}_2^3(\Omega_{(3)})\|^2+\|\mathbf{W}_t(\cdot,t);\ W_2^2(\Omega_{(3)})\|^2$$
$$+\|\mathbf{W}_{tt}(\cdot,t);\ W_2^1(\Omega_{(3)})\|^2\Big)\,dt\le c\sum_{j=1}^J\Big[\|F_j;\ W_2^2(0,T)\|^2 \tag{2.19}$$
$$+\|u_{02}^{(j)};\ W_2^3(\sigma_j)\|^2+\|f_2^{(j)};\ L_2(\Sigma_j^T)\|^2+\|f_{2t}^{(j)};\ L_2(\Sigma_j^T)\|^2\Big]:=c\,A_1,$$

Let us put

$$\mathbf{V}(x,t)=\mathbf{U}(x,t)+\mathbf{W}(x,t). \tag{2.20}$$

Then,

$$\operatorname{div}\mathbf{V}(x,t)=0,\ \ \mathbf{V}(x,t)\big|_{\partial\Omega}=0,\ \ \int_{\sigma_j}\mathbf{V}(x,t)\cdot\mathbf{n}(x)\,ds=F_j(t),\ j=1,\dots,J,$$

and for $x\in\Omega_j\setminus\Omega_{j3}$ the vector-function $\mathbf{V}(x,t)$ coincides with the velocity part $\mathbf{U}^{(j)}(x,t)$ of the corresponding nonstationary Poiseuille solution.

3. Navier-Stokes problem

3.1. Existence of the solution

Let us consider in the domain Ω Navier-Stokes problem (1.1). Assume that the initial velocity $\mathbf{u}_0$ and the external force $\mathbf{f}$ admit the representations (1.2). We look for the solution $(\mathbf{u},p)$ of (1.1) in the form

$$\mathbf{u}(x,t)=\mathbf{v}(x,t)+\mathbf{V}(x,t),\qquad p(x,t)=\widetilde{p}(x,t)+P(x,t), \tag{3.1}$$

where $\mathbf{V}$ is defined by (2.21) and P by (2.17). For $(\mathbf{v},\widetilde{p})$ we derive the following problem

$$\begin{cases}\mathbf{v}_t-\nu\Delta\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{v}+(\mathbf{V}\cdot\nabla)\mathbf{v}+(\mathbf{v}\cdot\nabla)\mathbf{V}+\nabla\widetilde{p}=\widetilde{\mathbf{f}},\\ \operatorname{div}\mathbf{v}(x,t)=0,\\ \mathbf{v}(x,t)\big|_{\partial\Omega}=0,\qquad \mathbf{v}(x,0)=\widetilde{\mathbf{u}}_0(x),\\ \int_{\sigma_j}\mathbf{v}(x,t)\cdot\mathbf{n}(x)\,ds=0,\qquad j=1,\dots,J,\end{cases} \tag{3.2}$$

where $\widetilde{\mathbf{u}}_0(x)=\widehat{\mathbf{u}}_0(x)-\mathbf{W}(x,0)$, $\widetilde{\mathbf{f}}(x,t)=\widehat{\mathbf{f}}(x,t)+\mathbf{f}^{(1)}(x,t)+\mathbf{f}^{(2)}(x,t)$,

$$\mathbf{f}^{(1)}(x,t)=(0,\ f_2^{(1)}(x,t))=\sum_{j=1}^J\big(\nu\zeta''(x_2^{(j)})(0,\,U_2^{(j)}(x_1^{(j)},t))$$
$$-\zeta(x_2^{(j)})\zeta'(x_2^{(j)})(0,\ |U_2^{(j)}(x_1^{(j)},t)|^2)-\zeta'(x_2^{(j)})x_2^{(j)}(0,\ q^{(j)}(t))\big), \tag{3.3}$$
$$\mathbf{f}^{(2)}(x,t)=-\mathbf{W}_t(x,t)+\nu\Delta\mathbf{W}(x,t)-(\mathbf{W}(x,t)\cdot\nabla)\mathbf{W}(x,t)$$
$$-(\mathbf{U}(x,t)\cdot\nabla)\mathbf{W}(x,t)-(\mathbf{W}(x,t)\cdot\nabla)\mathbf{U}(x,t).$$

Deriving (3.3) we have used that the pairs $\big(\mathbf{U}^{(j)}(x,t), P^{(j)}(x,t)\big), j = 1,\dots,J$, satisfy in Ω_j Navier–Stokes system with the right-hand side $\big(0, f_2^{(j)}(x_1^{(j)},t)\big)$ and the initial data $(0, u_{02}^{(j)}(x_1^{(j)}))$. By construction $\operatorname{supp}_x \mathbf{W}(x,t) \subset \Omega_{(3)}$. Therefore,

$$\operatorname{supp}_x \left[\mathbf{f}^{(1)}(x,t) + \mathbf{f}^{(2)}(x,t)\right] \subset \overline{\Omega}_{(3)}. \tag{3.4}$$

Using Sobolev imbedding theorems and (2.16), (2.20) we obtain the inequalities

$$\int_0^t \int_\Omega \sum_{k=1}^2 \left[|\mathbf{f}^{(k)}(x,\tau)|^2 + |\mathbf{f}_\tau^{(k)}(x,\tau)|^2\right] dx\, d\tau \le c\, A_2, \tag{3.5}$$

where A_2 depends quadratically on A_1 and does not depend on t (A_1 is defined in (2.20)).

Weak solution of problem (3.2) is a vector-function $\mathbf{v}$ belonging to the class

$$\mathcal{M}_1 = \left\{ \mathbf{v}: \ \operatorname{div} \mathbf{v} = 0, \ \mathbf{v}\big|_{\partial\Omega} = 0, \ \mathbf{v}, \ \nabla\mathbf{v}, \ \mathbf{v}_t, \ \nabla\mathbf{v}_t \in L_2(Q^T) \right\}$$

and satisfying the initial condition $\mathbf{v}(x,0) = \widetilde{\mathbf{u}}_0(x)$ and the integral identity

$$\begin{aligned} &\textstyle\int_0^t \int_\Omega \mathbf{v}_t \cdot \boldsymbol{\eta}\, dxd\tau + \nu \int_0^t \int_\Omega \nabla\mathbf{v} \cdot \nabla\boldsymbol{\eta}\, dxd\tau + \int_0^t \int_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{v}\cdot\boldsymbol{\eta}\, dxd\tau \\ &+ \textstyle\int_0^t \int_\Omega \big(\mathbf{V}\cdot\nabla\big)\mathbf{v}\cdot\boldsymbol{\eta}\, dxd\tau + \int_0^t \int_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{V}\cdot\boldsymbol{\eta}\, dxd\tau = \int_0^t \int_\Omega \widetilde{\mathbf{f}}\cdot\boldsymbol{\eta}\, dxd\tau \end{aligned} \tag{3.6}$$

for every $\boldsymbol{\eta} \in \mathcal{M}_2 = \left\{ \boldsymbol{\xi}: \ \operatorname{div}\boldsymbol{\xi} = 0, \ \boldsymbol{\xi}\big|_{\partial\Omega} = 0, \ \boldsymbol{\xi}, \ \nabla\boldsymbol{\xi} \in L_2(Q^T) \right\}.$

Theorem 3.1. *Let* $\partial\Omega \in C^4$, $\operatorname{div}\mathbf{u}_0 = 0$, $\mathbf{u}_0\big|_{\partial\Omega} = 0$, *and let* $\mathbf{u}_0$, $\mathbf{f}$ *admit representations* (1.2) *with* $u_{02}^{(j)} \in W_2^3(\sigma_j)$, $\widehat{\mathbf{u}}_0 \in W_2^2(\Omega)$, $f_2^{(j)}$, $f_{2t}^{(j)} \in L_2(\Sigma_j^T)$, $\widehat{\mathbf{f}}$, $\widehat{\mathbf{f}}_t \in L_2(Q^T)$, $T \in (0,\infty]$. *Moreover, let* $F_j(t) \in W_2^2(0,T)$ *and* $\big(u_{02}^{(j)}(x_1^{(j)}), f_2^{(j)}(x_1^{(j)},t)\big)$, $j = 1,\dots,J$, *satisfy the compatibility conditions* (1.3). *Then there exists a solution* $\mathbf{v} \in \mathcal{M}_1$ *of problem* (3.2) *satisfying the estimates*

$$\begin{aligned} &\sup_{t\in[0,T]} \Big(\|\mathbf{v}(\cdot,t);\ W_2^2(\Omega)\|^2 + \|\mathbf{v}_t(\cdot,t);\ L_2(\Omega)\|^2\Big) \\ &+ \textstyle\int_0^T \Big(\|\mathbf{v}(\cdot,t);\ W_2^2(\Omega)\|^2 + \|\mathbf{v}_t(\cdot,t);\ W_2^1(\Omega)\|^2\Big)\, dt \le c\, A_3, \\ &\textstyle\int_0^T \displaystyle\sup_{x\in\overline{\Omega}} \big(|\mathbf{v}(x,t)|^2\big)\, dt + \sup_{t\in[0,T]} \sup_{x\in\overline{\Omega}} \big(|\mathbf{v}(x,t)|^2\big) \le c\, A_3, \end{aligned} \tag{3.7}$$

where A_3 *depends only on norms of the data of the problem. Moreover, there exists a pressure function* $\widetilde{p}(x,t)$ *with* $\nabla\widetilde{p}(\cdot,t) \in L_2(\Omega)$ *such that the pair* $(v, \widetilde{p})$ *satisfies the system* (3.2) *almost everywhere in* Q^T. *There holds the estimate*

$$\sup_{t\in[0,T]} \|\nabla\widetilde{p}(\cdot,t);\ L_2(\Omega)\|^2 + \int_0^T \|\nabla\widetilde{p}(\cdot,t);\ L_2(\Omega)\|^2)\, dt \le c\, A_3. \tag{3.8}$$

The existence of a weak solution $\mathbf{v}$ to problem (3.2) could be proved using Galerkin approximations just in the same way as in the classical book of Ladyzhenskaya [5]. The essential part in this proof takes the derivation of a priori estimates of the solution (see [5], Chapter VI). Galerkin approximation $\mathbf{v}^{(m)}$ satisfy the following relations[1]

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega |\mathbf{v}(x,t)|^2\,dx + \nu\int_\Omega |\nabla\mathbf{v}(x,t)|^2\,dx \\ = -\int_\Omega \big(\mathbf{v}(x,t)\cdot\nabla\big)\mathbf{V}(x,t)\cdot\mathbf{v}(x,t)\,dx + \int_\Omega \widetilde{\mathbf{f}}(x,t)\cdot\mathbf{v}(x,t)\,dx, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \tfrac{1}{2}\tfrac{d}{dt}\int_\Omega |\mathbf{v}_t(x,t)|^2\,dx + \nu\int_\Omega |\nabla\mathbf{v}_t(x,t)|^2\,dx \\ = -\int_\Omega \big(\mathbf{v}_t\cdot\nabla\big)\mathbf{v}\cdot\mathbf{v}_t\,dx - \int_\Omega \big(\mathbf{V}_t\cdot\nabla\big)\mathbf{v}\cdot\mathbf{v}_t\,dx - \int_\Omega \big(\mathbf{v}_t\cdot\nabla\big)\mathbf{V}\cdot\mathbf{v}_t\,dx \\ - \int_\Omega \big(\mathbf{v}\cdot\nabla\big)\mathbf{V}_t\cdot\mathbf{v}_t\,dx + \int_\Omega \widetilde{\mathbf{f}}_t\cdot\mathbf{v}_t\,dx = \sum_{j=1}^{4} I_j + \int_\Omega \widetilde{\mathbf{f}}_t\cdot\mathbf{v}_t\,dx. \end{aligned} \tag{3.10}$$

Estimating the right-hand sides of (3.9), (3.10) with the help of (2.16), (2.20), (3.5) and applying the Gronwall inequality we derive the estimates

$$\begin{aligned} \int_\Omega |\mathbf{v}(x,t)|^2\,dx + \nu\int_0^t\int_\Omega |\nabla\mathbf{v}(x,\tau)|^2\,dx\,d\tau \le c\,A_3, \\ \int_\Omega |\mathbf{v}_t(x,t)|^2\,dx + \nu\int_0^t\int_\Omega |\nabla\mathbf{v}_\tau(x,\tau)|^2\,dx\,d\tau \le c\,A_3, \end{aligned} \tag{3.11}$$

The constant c in (3.11) does not depend on $t\in(0,T]$. Inequalities (3.11) give us the possibility to pass to a limit in the integral identity (3.6) for Galerkin approximation $\mathbf{v}^{(m)}$ and to prove that the sequence $\{\mathbf{v}^{(m)}\}$ converges to a weak solution $\mathbf{v}\in\mathcal{M}_1$ of problem (3.2) (see [5], Ch. VI for details). Inequalities (3.11) obviously remain valid for the limit function $\mathbf{v}$. Norms of the second derivatives of $\mathbf{v}$ and norms of $\nabla\tilde{p}$ could be estimated considering $\mathbf{v}$ as a solution to the stationary nonlinear Navier–Stokes problem with the right-hand side $\tilde{\mathbf{f}}-\mathbf{v}_t$ and using the known results concerning the stationary Navier–Stokes equations.

3.2. Weighted estimates of the solution

There holds the following

Theorem 3.2. *Assume that there hold the conditions of Theorem* 3.1 *and let, in addition,* $\widehat{\mathbf{u}}_0\in\mathcal{W}^1_{2,\boldsymbol{\beta}}(\Omega)$, $\widehat{\mathbf{f}}\in\mathcal{L}_{2,\boldsymbol{\beta}}(Q^T)$, $\beta_j\ge 0$, $j=1,\dots,J$, $T\in(0,\infty]$. *Then the weak solution* $\mathbf{v}$ *of problem* (3.2) *admits the estimate*

$$\begin{aligned} \sup_{t\in[0,T]} \|\mathbf{v}(\,\cdot\,,t);\ \mathcal{W}^1_{2,\boldsymbol{\beta}}(\Omega)\|^2 \\ + \int_0^T \big(\|\mathbf{v}(\,\cdot\,,t);\ \mathcal{W}^1_{2,\boldsymbol{\beta}}(\Omega)|^2 + \|\mathbf{v}_t(\,\cdot\,,t);\ \mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)|^2\big)\,dt \le c\,A_4, \end{aligned} \tag{3.12}$$

where A_4 *depends on weighted norms of* $\widehat{\mathbf{u}}_0$ *and* $\widehat{\mathbf{f}}$.

In order to prove (3.12) we first multiply the system (3.2) by $\mathbf{v}(x,t)E^{(k)}_{\boldsymbol{\beta}}(x)+\mathbf{W}^{(k)}(x,t)$, where $E^{(k)}_{\boldsymbol{\beta}}$ is a step weight function defined by (2.3) and $\mathbf{W}^{(k)}$ is

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[1] Below the index "m" is omitted

the solution of the equation (2.6) constructed in Lemma 2.1. Then $\operatorname{div}\big(\mathbf{v}E_{\boldsymbol{\beta}}^{(k)}+\mathbf{W}^{(k)}\big)=0$ and integrating by parts in Ω we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\mathbf{v}(x,t)|^2dx+\nu\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2dx\\ =\textstyle\int_\Omega\widetilde{\mathbf{f}}\cdot\big(\mathbf{v}E_{\boldsymbol{\beta}}^{(k)}+\mathbf{W}^{(k)}\big)dx-\int_\Omega\mathbf{v}_t\cdot\mathbf{W}^{(k)}dx-\nu\int_\Omega\nabla\mathbf{v}\cdot\nabla E_{\boldsymbol{\beta}}^{(k)}\cdot\mathbf{v}\,dx\\ -\nu\textstyle\int_\Omega\nabla\mathbf{v}\cdot\nabla\mathbf{W}^{(k)}dx-\int_\Omega\big((\mathbf{v}+\mathbf{V})\cdot\nabla\big)\mathbf{v}\cdot\mathbf{v}E_{\boldsymbol{\beta}}^{(k)}dx\\ -\textstyle\int_\Omega\big((\mathbf{v}+\mathbf{V})\cdot\nabla\big)\mathbf{v}\cdot\mathbf{W}^{(k)}dx-\int_\Omega\big(\mathbf{v}\cdot\nabla\big)\mathbf{V}\cdot\big(\mathbf{v}E_{\boldsymbol{\beta}}^{(k)}+\mathbf{W}^{(k)}\big)\,dx.\end{aligned}\tag{3.13}$$

Note that all integrals in (3.13) are finite since the function $E_{\boldsymbol{\beta}}^{(k)}(x)$ is equal to a constant for large $|x|$ and $\mathbf{W}^{(k)}(x,t)$ has a compact support contained in $\overline{\Omega}_{(k)}$. Estimating the right-hand side of (3.13) we obtain

$$\begin{aligned}\tfrac{1}{2}\tfrac{d}{dt}\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\mathbf{v}(x,t)|^2\,dx+\nu\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2\,dx\\ \le c\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,t)|^2\,dx+\gamma_*\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\mathbf{v}_t(x,t)|^2\,dx\\ +\big(c_1\gamma_*+\tfrac{\nu}{2}\big)\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2\,dx+c\,\mathcal{K}(t)\int_\Omega E_{\beta}^{(k)}(x)|\mathbf{v}(x,t)|^2\,dx,\end{aligned}\tag{3.14}$$

where γ_* is the constant from (2.1_3) and

$$\mathcal{K}(t)=\sup_{x\in\overline{\Omega}}\Big[|\mathbf{V}(x,t)|^2+|\nabla\mathbf{V}(x,t)|^2+|\mathbf{v}(x,t)|^2\Big].\tag{3.15}$$

Second, we multiply (3.2) by $\mathbf{v}_t(x,t)E_{\boldsymbol{\beta}}^{(k)}(x)+\mathbf{W}_t^{(k)}(x,t)$ and integrate by parts in Ω:

$$\begin{aligned}\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\mathbf{v}_t(x,t)|^2\,dx+\tfrac{\nu}{2}\tfrac{d}{dt}\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2\,dx\\ =-\textstyle\int_\Omega\mathbf{v}_t(x,t)\mathbf{W}_t^{(k)}(x,t)\,dx-\nu\int_\Omega\nabla\mathbf{v}(x,t)\cdot\nabla E_{\boldsymbol{\beta}}^{(k)}(x)\cdot\mathbf{v}_t(x,t)\,dx\\ -\nu\textstyle\int_\Omega\nabla\mathbf{v}(x,t)\cdot\nabla\mathbf{W}_t^{(k)}(x,t)\,dx-\int_\Omega\big((\mathbf{v}+\mathbf{V})\cdot\nabla\big)\mathbf{v}\cdot\mathbf{v}_tE_{\boldsymbol{\beta}}^{(k)}\,dx\\ -\textstyle\int_\Omega\big((\mathbf{v}+\mathbf{V})\cdot\nabla\big)\mathbf{u}\cdot\mathbf{W}_t^{(k)}\,dx-\int_\Omega\big(\mathbf{v}\cdot\nabla\big)\mathbf{V}\cdot\big(\mathbf{v}_tE_{\boldsymbol{\beta}}^{(k)}+\mathbf{W}_t^{(k)}\big)\,dx\\ +\textstyle\int_\Omega\widetilde{\mathbf{f}}(x,t)\cdot\big(\mathbf{v}_t(x,t)E_{\boldsymbol{\beta}}^{(k)}(x)+\mathbf{W}_t^{(k)}(x,t)\big)\,dx.\end{aligned}\tag{3.16}$$

From (3.16) we get that

$$\begin{aligned}\tfrac{\nu}{2}\tfrac{d}{dt}\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2\,dx+\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\mathbf{v}_t(x,t)|^2\,dx\\ \le\big(c_2\gamma_*+\tfrac{3}{8}\big)\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\mathbf{v}_t(x,t)|^2\,dx+c\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,t)|^2\,dx\\ +c\gamma_*\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2\,dx+c\,\mathcal{K}(t)\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\nabla\mathbf{v}(x,t)|^2\,dx.\end{aligned}\tag{3.17}$$

If the constant γ_* is sufficiently small (see the property (2.1_3) of the weight function $E_{\boldsymbol{\beta}}$), from (3.14), (3.17) follows the inequality

$$\begin{aligned}\tfrac{d}{dt}\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)\big(|\mathbf{v}(x,t)|^2+\nu|\nabla\mathbf{v}(x,t)|^2\big)\,dx\\ \le c_3\,\mathcal{K}(t)\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)\big(|\mathbf{v}(x,t)|^2+\nu|\nabla\mathbf{v}(x,t)|^2\big)\,dx\\ +c_4\textstyle\int_\Omega E_{\boldsymbol{\beta}}^{(k)}(x)|\widetilde{\mathbf{f}}(x,t))|^2\,dx.\end{aligned}\tag{3.18}$$

Using Gronwall's lemma from (3.18), (3.14) and (3.17) we derive

$$\begin{aligned} &\int_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)\big(|\mathbf{v}(x,t)|^2+\nu|\nabla\mathbf{v}(x,t)|^2\big)\,dx \\ +&\int_0^T\int_\Omega E^{(k)}_{\boldsymbol{\beta}}(x)\big(\nu|\nabla\mathbf{v}(x,t)|^2+|\mathbf{v}_t(x,t)|^2\big)\,dxdt\le c_5\,A_4 \end{aligned} \tag{3.19}$$

with the constant c_5 independent of k. Since $E^{(k)}_{\boldsymbol{\beta}}(x)=E_{\boldsymbol{\beta}}(x)$ for $x\in\Omega_{(k)}$, from (3.19) follows the estimate

$$\begin{aligned} &\int_{\Omega_{(k)}} E_{\boldsymbol{\beta}}(x)\big(|\mathbf{v}(x,t)|^2+\nu|\nabla\mathbf{v}(x,t)|^2\big)\,dx \\ +\tfrac{1}{2}&\int_0^T\int_{\Omega_{(k)}} E_{\boldsymbol{\beta}}(x)\big(\nu|\nabla\mathbf{v}(x,t)|^2+|\mathbf{v}_t(x,t)|^2\big)\,dxdt\le c_5\,A_4 \end{aligned} \tag{3.20}$$

and passing $k\to\infty$ we get (3.12).

Remark 3.3. In the case of an exponential weight function, i.e., in the case where $E_{\beta_j}(x)=\exp\big(2\beta_j x_2^{(j)}\big)$, the condition that γ_* in (2.1$_3$) is sufficiently small could be satisfied, if we assume that $\beta_*=\max\limits_{j=1,\dots,J}\beta_j$ is sufficiently small. Thus, in the case of exponential weight function we have a restriction on the on weight exponents $\beta_j,\ j=1,\dots,J$. In the case of the power weight function, i.e., if $E_{\beta_j}(x)=\big(1+\delta^2|x_2^{(j)}|^2\big)^{\beta_j}$, this condition could be satisfied taking sufficiently small δ. Since for different δ norms in the space $\mathcal{L}_{2,\boldsymbol{\beta}}(\Omega)$ are equivalent, the results of Theorem 3.2 are true in this case without any restriction on weight exponents $\beta_j,\ j=1,\dots,J$.

Remark 3.4. From Theorem 3.2 it follows that the decay rate of the perturbation $\mathbf{v}(x,t)$ of the flux carrier $\mathbf{V}(x,t)$ depends only on the decay rate of $\widehat{\mathbf{f}}(x,t)$ and $\widehat{\mathbf{u}}_0(x)$. If $\widehat{\mathbf{f}}(x,t)=0$, $\widehat{\mathbf{u}}_0(x)=0$, then the functions $\widetilde{\mathbf{f}}(x,t)$ and $\widetilde{\mathbf{u}}_0(x)$ have compact supports and, therefore, belong to weighted function spaces with exponential weight function. Thus, we conclude that in this case $\mathbf{u}(x,t)$ tends in each outlet to infinity Ω_j to the corresponding nonstationary Poiseuille flow $\mathbf{U}^{(j)}(x,t)$ exponentially as $|x|\to\infty$. This is true for arbitrary large fluxes $F_j(t)$ and an infinite time interval.

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A Link between Local Solvability and Partial Analyticity of Several Classes of Degenerate Parabolic Operators

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Abstract. The aim of this work is to find a necessary and sufficient condition for local solvability of some classes of degenerate parabolic operators. The conditions are imposed on the right-hand side f of the corresponding equation. It is well known that the operators under consideration are nonsolvable for a "massive" set of smooth functions f.

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1. This paper deals with the link between the local solvability and the partial analyticity of the right-hand side f of the partial differential equation $Pu = f$, P being a degenerate second order parabolic operator. The operator P is locally nonsolvable in the space of Schwartz distributions $\mathcal{D}'$. Historically, it was at first the famous example of Lewy (see [4, 8, 1]) of locally nonsolvable first order PDO that stimulated the creation and further development of the theory of local (non)solvability. Due to Hörmander [5] the proof of local nonsolvability was reduced to the violation of some a priori estimates satisfied by the operator under consideration P, while the positive results were proved on the basis of a priori estimates fulfilled by P in appropriate Sobolev spaces. Certainly, it is very interesting to find necessary and sufficient conditions for the local solvability imposed on the right-hand side f. Such type condition was given by Lewy in [8]. The necessary and sufficient condition for the local solvability of Lewy's equation $\frac{\partial u}{\partial z} + i\overline{z}\frac{\partial u}{\partial t} = f$ was found in [4]. If $\mathbf{R}^3$ is realized as the boundary of the generalized "upper half-space" in $\mathbf{C}^2$, then the conditions are, near a point $Q \in \mathbf{R}^3$, the analytic continuability of the Cauchy-Szegö integral of f past Q. The nonsolvable Mizohata operator was studied in details by Ninomiya in [11]. Concerning general principal type operators conditions characterizing admissible data in terms of appropriate projectors were given by several authors but we shall mention here the paper [12] only. Interesting results on the same subject for nonsolvable operators with multiple symplectic characteristics are given in [2]. Many examples and comments on nonsolvable partial differential operators can be found in [3, 9].

In our paper we deal with the nonsolvable parabolic operator $Mu = \partial_{x_1} u + x_1^{2p+1} \partial_{x_2}^2 u = f$ and we prove the existence of (partial) analytic type conditions with respect to the space variable x_2 which are satisfied by f. To do this we shall use the Green function of the Dirichlet problem for the heat equation in a half-strip ([7, 13]).

2. We shall formulate here our main result.

So consider the polynomial $p(y; \xi_1, \eta)$, $x_1 \in \mathbf{R}^1$, $\xi_1 \in \mathbf{R}^1$, $y, \eta \in \mathbf{R}^n$. Assume that there exist constants $\sigma_j > 0$, $j = 1, \dots, n$; $\mu > 0$, r and such that $p(\lambda^{-\sigma} y; \lambda^{\mu} \xi_1, \lambda^{\sigma} \eta) = \lambda^r p(y; \xi_1, \eta)$, $\forall \lambda > 0$, $\forall (y; \xi_1, \eta)$, where

$$\lambda^{-\sigma} y = (\lambda^{-\sigma_1} y_1, \dots, \lambda^{-\sigma_n} y_n).$$

The polynomial p is called quasihomogeneous.

In [3] the following result was proved.

Theorem. *Consider the PDO: $P(y; D_{x_1}, D_y)$ and suppose that the symbol $p(y; \xi_1, \eta)$ is quasihomogeneous, $\max_j \frac{\sigma_j}{\mu} < 1$ and that one can find a non-flat at the origin function $u(y) \in Ker P(y; 1, D_y) \cap \mathcal{S}(\mathbf{R}^n)$ $(u \in Ker P(y; -1, D_y) \cap \mathcal{S}(\mathbf{R}^n))$.*

Then the L_2 adjoint operator P^* of P is locally nonsolvable at the origin 0 in the Schwartz distribution space $\mathcal{D}'$.

Remark 1. $\mathcal{S}(\mathbf{R}^n)$ stands for the space of Schwartz rapidly decreasing at infinity functions and "non-flat" means the existence of a multi-index α_0 having the following property: $D^{\alpha_0} u(0) \neq 0$, $D^{\alpha} u(0) = 0$, $|\alpha| < \alpha_0$.

We shall illustrate this theorem with several simple examples.

Example 1. Consider in $\mathbf{R}^2$ the operator $P_\pm = D_y \pm i y^l D_x^m$. Then $P_\pm$ is quasihomogeneous with weights $\sigma_1 = 1$, $\mu = \frac{l+1}{m}$. It is well known that $P_\pm$ is C^∞ hypoelliptic iff at least one of the integers l, m is even. If l, m are odd and $\frac{l+1}{m} > 1$, then $P_\pm$ is locally nonsolvable at 0 in $\mathcal{D}'$.

Example 2. Consider now the operator P_+ for l odd, m even, $\frac{l+1}{m} > 1$. Then P_+ is locally nonsolvable at 0 in $\mathcal{D}'$. In the case $m = 2$ we have that the operator $iP_+ = \partial_y + y^l \partial_x^2$ is locally nonsolvable for $l > 1$ odd (see also [6], $l \geq 1$). For the sake of simplicity we shall write our operator in the following form:

$$Mu = \partial_{x_1} u + x_1^{2p+1} \partial_{x_2}^2 u = f(x_1, x_2)$$

and u is sufficiently smooth (say $C^3(\omega)$ in some neighborhood ω of 0). Evidently, $f(x_1, x_2) = \frac{f(x_1,x_2)+f(-x_1,x_2)}{2} + \frac{f(x_1,x_2)-f(-x_1,x_2)}{2}$.

Remark 2. After the change $x_1 \to -x_1$, $x_2 \to \pm x_2$ the operator M is transformed into $-M$.

We shall suppose further on that $f(x_1, x_2)$ is even with respect to x_1, i.e.,

$$f(-x_1, x_2) = f(x_1, x_2), \forall (x_1, x_2) \in \omega, \forall (-x_1, x_2) \in \omega. \tag{1}$$

Definition 3. The function $f(x_1, x_2)$ is called admissible if the equation $Mu = f$ possesses a classical solution $u \in C^2$ in ω.

The function f is supposed to be $C^2(\omega)$ smooth. One can easily see that the function f is admissible if and only if $x_1^{2p+1} \int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt$ is admissible.

In fact, if $Mu = f$, $u \in C^2(\omega)$, then we define $\nu = u - \int_0^{x_1} f(t, x_2)\, dt$ and we obtain that $M\nu = -x_1^{2p+1} \int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt$. Conversely, assume that $M\nu = x_1^{2p+1} \int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt$ for some $\nu \in C^2$. Put $u = -\nu + \int_0^{x_1} f(t, x_2)\, dt$. The integral $\int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt$ is odd with respect to x_1 as $f''_{x_2 x_2}(t, x_2)$ is even with respect to t. Therefore, $x_1^{2p+1} \int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt$ is even in x_1. Obviously, $Mu = -M\nu + f + x_1^{2p+1} \int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt = f$.

Put $f^{\#}(x_1, x_2) = \int_0^{x_1} f''_{x_2 x_2}(t, x_2)\, dt \Rightarrow f^{\#}(-x_1, x_2) = f^{\#}(x_1, x_2)$. Then the following proposition holds.

Proposition 4. *The function $f \in C^2(\omega)$, $f(-x_1, x_2) = f(x_1, x_2)$ is admissible iff $x_1^{2p+1} f^{\#}(x_1, x_2)$ is admissible.*

Let us split the solution u into even v and odd w parts with respect to x_1. This way we get:

$u = v + w \Rightarrow f = Mu = Mv + Mw$.

Certainly, $Mv = 0$, $Mw = f$.

Put $G(s, y; \tau, \xi) = G(s - \tau; y, \xi)$ and define

$$G(s, y; \tau, \xi) = \sum_{n=-\infty}^{\infty} [G_0(s, y; \tau, \xi - 2nl) - G_0(s, y; \tau, -\xi - 2nl)] \tag{2}$$

where $G_0(s - \tau; y, \xi) = G_0(s, y; \tau, \xi) = \frac{1}{2\sqrt{\pi(s-\tau)}} e^{-\frac{(y-\xi)^2}{4(s-\tau)}}$ for $s \geq \tau$ and $G_0 = 0$ otherwise.

Theorem 5. *Consider the degenerate parabolic operator $M = \partial_{x_1} + x_1^{2p+1} \partial^2_{x_2}$, $p \in \mathbf{N} \cup \{0\}$ and suppose that $f(x_1, x_2)$ satisfying* (1) *is admissible function, $(x_1, x_2) \in \omega_1$, $|x_1| \leq (mT)^{\frac{1}{m}}$, $m = 2p + 2$, for some $T > 0$ and $|x_2| \leq \frac{l}{2}$, for some $l > 0$. Then the function*

$$F_{T'}(x_2 + \frac{l}{2}) = \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^{T'} G(\nu, x_2 + \frac{l}{2}, \mu + \frac{l}{2}) f^{\#}((m\nu)^{\frac{1}{m}}, \mu)\, d\mu d\nu, \; 0 < T' < T.$$

can be prolonged analytically with respect to x_2 in the open parallelogram $Q = \{z = x_2 + i\theta\} \subset \mathbf{C}^1$ having the vertices $A(-\frac{l}{2}, 0)$, $B(0, -id)$, $C(\frac{l}{2}, 0)$, $D(0, id)$, $d = \text{const.} > 0$, d depends on T'.

Remark 6. According to (2)

$$G(\nu, x_2+\frac{l}{2}; \mu+\frac{l}{2}) = \frac{1}{2\sqrt{\pi\nu}}\sum_{n=-\infty}^{\infty}\left[e^{-\frac{(x_2-\mu+2nl)^2}{4\nu}} - e^{-\frac{(x_2+\mu+(2n+1)l)^2}{4\nu}}\right], \quad \nu \geq 0.$$

This is a slight generalization of Theorem 5.

Theorem 7. *Consider the equation $Mu = f$ in the rectangle $\omega = \{|x_1| \leq (mT)^{\frac{1}{m}}, |x_2| \leq \frac{l}{2}\}$ and assume that the function f is even with respect to x_1 and that $f^{\#}((mT)^{\frac{1}{m}}, \pm\frac{l}{2}) = 0$. Then f is admissible if and only if the equation $Pw_1 = 0$, where $P = \partial_s - \partial_y^2$, possesses a classical solution $w_1(s,y)$ in the rectangle $\omega_1 = \{0 \leq s \leq T, 0 \leq y \leq l\}$ and such that $w_1|_{s=T} = F_T(y) = \int_0^l\int_0^T G(\nu, y; \mu+\frac{l}{2})\Phi(\nu,\mu)\,d\mu d\nu$ and $\Phi(\nu,\mu) = f^{\#}((m\nu)^{\frac{1}{m}}, \mu)$.*

3. We propose now a small excursion in the theory of the famous θ-function [10] studied by Jacobi, Weierstrass, Fourier, Hermite. This analytic function in $z \in \mathbf{C}^1$, $\tau \in H = \{\Im\tau > 0\}$ is defined by the formula

$$\theta(z,\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n z}. \tag{3}$$

Then: $\theta(z+\tau,\tau) = e^{-\pi i\tau - 2\pi i z}\theta(z,\tau)$, $\theta(z+1,\tau) = \theta(z,\tau)$. Moreover, $\theta(z,\tau) = 0 \iff z = \frac{1}{2} + \frac{\tau}{2} + n + m\tau$; $m, n \in \mathbf{Z}$.

One can easily see that

$$\theta(x, it) = \sum_{-\infty}^{\infty} e^{-\pi n^2 t + 2\pi i n x} = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}\cos 2\pi n x,$$

$t > 0$ and therefore,

$$\frac{\partial}{\partial t}\theta(x,it) = \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\theta(x,it),\ t > 0.$$

According to the functional equation of θ [10]: $\theta(\frac{z}{\tau}, -\frac{1}{\tau}) = e^{-\frac{i\pi}{4}}\tau^{1/2}e^{\frac{\pi i z^2}{\tau}}\theta(z,\tau)$, applied for $z = x$, $\tau = it$, $t > 0$, we have:

$$\theta\left(\frac{x}{it}, \frac{i}{t}\right) = t^{1/2}e^{\frac{\pi x^2}{t}}\theta(x,it) \Rightarrow 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 t}\cos 2\pi n x$$

$$= t^{-1/2}e^{-\frac{\pi x^2}{t}}\sum_{-\infty}^{\infty} e^{-\frac{\pi n^2}{t} + \frac{2\pi n x}{t}} = t^{-1/2}e^{-\frac{\pi x^2}{t}}\sum_{n=-\infty}^{\infty} e^{-\pi\frac{(x-n)^2 - x^2}{t}}$$

$$= t^{-1/2}\sum_{n=-\infty}^{\infty} e^{-\frac{\pi(x-n)^2}{t}}.$$

This way we obtain the famous Poisson summation formula

$$\theta(x,it) = 1 + 2\sum_{1}^{\infty}\cos 2\pi n x e^{-\pi n^2 t} = t^{-1/2}\sum_{-\infty}^{\infty} e^{-\frac{\pi(x-n)^2}{t}},\ t > 0. \tag{4}$$

Suppose that $f \in C^\infty(\Pi^1)$, Π^1 being the unit circle (one dimensional torus $\mathbf{R}/\mathbf{Z}$). Its Fourier series has the form

$$f(x) = \sum_{-\infty}^{\infty} a_m e^{2\pi i m x}, \ 0 < x < 1$$

and therefore

$$\int_0^1 \theta(x, it) f(x)\, dx = \sum_{-\infty}^{\infty} a_{-n} e^{-\pi n^2 t}, \quad \text{i.e.,}$$

$$\lim_{t \to +0} \int_0^1 \theta(x, it) f(x)\, dx = \sum_{-\infty}^{\infty} a_n = f(0).$$

Thus we conclude that $\theta(x, it)$ is a fundamental solution of the heat equation $w_t = \frac{1}{4\pi} w_{xx}$ on the circle in x. Evidently, $\theta(0, it) = \theta(1, it)$, $\forall t > 0$.

4. We are going to prove Theorem 5. The proof will be split into several parts.

a) We make the following change of variables in the equation $Mu = f$:

$$\left| \begin{array}{l} t = \frac{x_1^m}{m} > 0, \\ \\ x = x_2 \end{array} \right. , \ m = 2p+2, \quad i.e., \quad \left| \begin{array}{l} x_1 = (mt)^{\frac{1}{m}} > 0 \\ \\ x_2 = x \end{array} \right. ,$$

where $0 < t \le T$, $|x| \le \frac{l}{2}$.

Then our operator M takes the form $M = x_1^{2p+1} L$ and $L = \frac{\partial}{\partial t} + \frac{\partial^2}{\partial x^2}$ is the backward parabolic operator. Thus, $Lv = 0$, $x_1^{2p+1} Lw = f((mt)^{\frac{1}{m}}, x)$.

Having in mind the fact that L is Gevrey G_2 hypoelliptic in t and analytic hypoelliptic in x we conclude that for each $t > 0$ fixed the function $v(t, x)$ is analytic with respect to x. According to Proposition 4 and without loss of generality we can take $f = x_1^{2p+1} f^\#(x_1, x_2)$ and consequently $Lw(t, x) = f^\#((mt)^{\frac{1}{m}}, x)$ in $\tilde{w} = \{(t, x) : 0 < t \le T, |x| \le \frac{l}{2}\}$ for some $T > 0$ and $l > 0$.

Remark 8. As $t^{\frac{1}{m}} \in C^{\frac{1}{m}}$, i.e., is Hölder continuous with exponent $\frac{1}{m}$, we conclude that $f^\#((mt)^{\frac{1}{m}}, x)$ is Hölder continuous only.

b) The classical function $w \in C^0$ is odd in $x_1 \Rightarrow w(0, x_2) = 0 \Rightarrow w(0, x) = 0$, $|x| \le \frac{l}{2}$. The function w satisfies the following mixed problem for the backward parabolic operator L:

$$\left| \begin{array}{l} Lw = \Phi(t, x) = f^\#((mt)^{\frac{1}{m}}, x) \\ w|_{t=T} = \varphi(x), \ -\frac{l}{2} \le x \le \frac{l}{2} \\ w|_{x=-\frac{l}{2}} = g_1(t), \ w|_{x=\frac{l}{2}} = g_2(t), \ 0 \le t \le T, \end{array} \right. \tag{5}$$

$w(0, x) = 0$ and the compatibility conditions $\varphi(-\frac{l}{2}) = g_1(T)$, $\varphi(\frac{l}{2}) = g_2(T)$.

From now on we shall work everywhere writing T instead of T'.

Let us make now the change of variables in $\tilde{w}$:

$$\left| \begin{array}{l} s = T - t \geq 0 \\ \\ y = x + \frac{l}{2} \in [0, l] \end{array} \right., \quad \text{i.e.} \quad , \left| \begin{array}{l} t = T - s, s \geq 0, s \leq T \\ \\ x = y - \frac{l}{2}, y \in [0, l] \end{array} \right.$$

and

$$t = T \iff s = 0; t = 0 \iff s = T,\ x = -\frac{l}{2} \iff y = 0,\ x = \frac{l}{2} \iff y = l.$$

Then the function $z(s, y) = w(t, x)$, $\tilde{\Phi}(s, y) = \Phi(t, x) = \Phi(T - s, y - \frac{l}{2})$, verifies in the infinite rectangle $\{s \geq 0, 0 \leq y \leq l\}$ the mixed problem:

$$\left| \begin{array}{l} -\frac{\partial z}{\partial s} + \frac{\partial^2 z}{\partial y^2} = \tilde{\Phi}(s, y) \\ z|_{s=0} = \tilde{\varphi}(y) \equiv \varphi(y - \frac{l}{2}) \in C[0, l] \\ z|_{y=0} = \tilde{g}_1(s) = g_1(T - s) \in C(s \geq 0) \\ z|_{y=l} = \tilde{g}_2(s) = g_2(T - s) \in C(s \geq 0), \end{array} \right. \tag{6}$$

and the compatibility condition $\tilde{g}_2(0) = \tilde{\varphi}(l)$, $\tilde{g}_1(0) = \tilde{\varphi}(0)$. Evidently, $w|_{t=0} = 0 \Rightarrow z|_{s=T} = 0$.

As z is the classical (unique) solution of the Dirichlet problem in $\{s \geq 0, 0 \leq y \leq l\}$ to the heat equation $Pz = \frac{\partial z}{\partial s} - \frac{\partial^2 z}{\partial y^2} = -\tilde{\Phi}(s, y)$, we can apply the Green formula expressing z by $\tilde{\Phi}$, $\tilde{\varphi}$, $\tilde{g}_1$, $\tilde{g}_2$. The validity of the integral representation of z proposed below, can be found in [7, 13]. Moreover, according to Krylov [7], more precise results on the solvability of (6) in appropriate Hölder classes with two weights $C^{\alpha, \frac{\alpha}{2}}(\overline{D})$ are true.

So

$$\begin{aligned} z(s, y) = & \int_0^s \left. \frac{\partial G}{\partial \xi} \right|_{\xi=0} \tilde{g}_1(\tau)\, d\tau - \int_0^s \left. \frac{\partial G}{\partial \xi} \right|_{\xi=l} \tilde{g}_2(\tau)\, d\tau \\ & + \int_0^l G(s, y; 0, \xi) \tilde{\varphi}(\xi)\, d\xi - \int_0^l \int_0^s G(s, y; \tau, \xi) \tilde{\Phi}(\tau, \xi)\, d\xi d\tau \end{aligned} \tag{7}$$

and the Green function $G(s, y; \tau, \xi)$ is defined by the formula (2).

Remark 9. One can easily see that $G|_{y=0} = G|_{y=l} = 0$, $PG = \frac{\partial G}{\partial s} - \frac{\partial^2 G}{\partial y^2} = 0$; (τ, ξ) being parameters and $G(s, y; \tau, \xi)|_{s=\tau} = \delta(y - \xi)$.

We have that

$$G(s, y; 0, \xi) = G(s, y, \xi) = \frac{1}{2\sqrt{\pi s}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(y - \xi + 2nl)^2}{4s}} - e^{\frac{(y + \xi + 2nl)^2}{4s}} \right],\ s \geq 0.$$

This is uniformly and absolutely convergent series for $|n| \geq 1$, $s \geq 0$ and $0 < \varepsilon_1 \leq y \leq \varepsilon_2 < l$, $0 \leq \xi \leq l$ or: $0 \leq y \leq l$, $\varepsilon_1 \leq \xi \leq \varepsilon_2$. The only singular term is $e^{-\frac{(y-\xi)^2}{4s}}$, $s \geq 0$, $(n = 0)$ and its singularity is attained at $y = \xi$.

In order to investigate the analytic continuation with respect to x of the integral terms in the right-hand side of (7) we must write the corresponding derivatives $\frac{\partial G}{\partial \xi}|_{\xi=0}$, $\frac{\partial G}{\partial \xi}|_{\xi=l}$.

Simple calculations show that

$$\frac{\partial G}{\partial \xi}|_{\xi=0} = \frac{2}{\sqrt{\pi s}} \sum_{n=-\infty}^{\infty} \frac{(y+2nl)}{4s} e^{-\frac{(y+2nl)^2}{4s}}. \tag{8}$$

As $0 \leq y \leq l$ the only singularity in (8) is given by $\frac{y}{s^{3/2}} e^{-\frac{y^2}{4s}}$, $y = s = 0$. The series over $|n| \geq 1$ is uniformly and absolutely convergent.

In a similar way we have:

$$\frac{\partial G}{\partial \xi}|_{\xi=l} = \frac{1}{\sqrt{\pi s}} \sum_{n=-\infty}^{\infty} \left[\frac{y+(2n-1)l}{4s} e^{-\frac{(y+(2n-1)l)^2}{4s}} + \frac{y+(2n+1)l}{4s} e^{-\frac{(y+l(2n+1))^2}{4s}} \right]. \tag{9}$$

The series is absolutely and uniformly convergent for $|n| \geq 2$, there are no singularities for $0 \leq y < l$, $s \geq 0$. The only singularity appears for $y = l$, $s = 0$ and can be written in the form: $\frac{y-l}{s} e^{-\frac{(y-l)^2}{4s}}$.

Going back to the coordinates (t, x) we obtain:

$$0 = z\left(T, x + \frac{l}{2}\right) \equiv z(T, y) = \int_0^T \frac{\partial G}{\partial \xi}\bigg|_{\xi=0, s=T} \tilde{g}_1(\tau)\, d\tau \tag{10}$$

$$- \int_0^T \frac{\partial G}{\partial \xi}\bigg|_{\xi=l, s=T} \tilde{g}_2(\tau)\, d\tau + \int_0^l G\left(T, x + \frac{l}{2}, \xi\right) \tilde{\varphi}(\xi)\, d\xi$$

$$- \int_0^l \int_0^T G\left(T - \tau, x + \frac{l}{2}, \xi\right) \tilde{\Phi}(\tau, \xi)\, d\xi d\tau \equiv I_1\left(x + \frac{l}{2}\right) + I_2\left(x + \frac{l}{2}\right)$$

$$+ I_3\left(x + \frac{l}{2}\right) + I_4\left(x + \frac{l}{2}\right).$$

Making the change:

$$\mu = \xi - \frac{l}{2}, \quad T - \tau = \nu \quad \text{in } I_4$$

we get:

$$I_4(x + \frac{l}{2}) = - \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_0^T G\left(\nu, x + \frac{l}{2}, \mu + \frac{l}{2}\right) \Phi(\nu, \mu)\, d\mu d\nu, \tag{11}$$

where $\Phi(\nu, \mu) = f^{\#}((m\nu)^{\frac{1}{m}}, \mu)$ and

$$G(\nu, x + \frac{l}{2}, \mu + \frac{l}{2}) = \frac{1}{2\sqrt{\pi \nu}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(x-\mu+2nl)^2}{4\nu}} - e^{-\frac{(x+\mu+(2n+1)l)^2}{4\nu}} \right].$$

Remark 10. The change $x_2 = x$ shows that $I_4(x_2 + \frac{l}{2})$ coincides with $-F_T(x_2 + \frac{l}{2})$ participating in the conditions of Theorem 5.

c) We will show now that I_1, I_2, I_3 can analytically be continued in y, i.e., in $x_2 = y - \frac{l}{2}$. Consider $III(y) = \int_0^l G(T, y, \xi)\tilde{\varphi}(\xi)\, d\xi$, $\tilde{\varphi} \in C[0, l]$. The corresponding series is absolutely and uniformly convergent with respect to $0 \le y \le l$, $0 \le \xi \le l$, as

$$G(T, y, \xi) = \frac{1}{2\sqrt{\pi T}} \sum_{n=-\infty}^{\infty} \left[e^{-\frac{(y-\xi+2nl)^2}{4T}} - e^{-\frac{(y+\xi+2nl)^2}{4T}} \right].$$

The series is analytic in $(y + i\theta)$ in the set $\{0 \le y \le l, |\theta| \le K\} \subset \mathbf{C}^1$ for each $K > 0$. In fact,

$$\left| e^{-\frac{(y-\xi+2nl+i\theta)^2}{4T}} \right| = e^{\frac{-(y-\xi+2nl)^2+\theta^2}{4T}}$$

and

$$|y - \xi + 2nl| \ge 2l(|n| - 1/2) \ge 0 \quad \text{for } |n| \ge 1.$$

Therefore, the series $G(T, y + i\theta, \xi)$ is uniformly and absolutely convergent in the above-mentioned strip. Similar considerations work for $\sum_{n=-\infty}^{\infty} e^{-\frac{(y+\xi+2nl+i\theta)^2}{4T}}$.

Conclusion: $III(x_2 + \frac{l}{2})$ is analytic in each rectangle $\{|x_2| \le \frac{l}{2}, |\theta| \le K, K = \text{const.} > 0\}$, i.e., in the strip $\{z = x_2 + i\theta, |x_2| < \frac{l}{2}\}$, $III = I_3$.

As we know

$$I_1 = I(y) = \int_0^T \frac{\partial G}{\partial \xi}(T - \tau, y, 0)\tilde{g}_1(\tau)\, d\tau,\ \tilde{g}_{1,2} \in C([0, T]),$$

$$-II_2 = II(y) = \int_0^T \frac{\partial G}{\partial \xi}(T - \tau, y, l)\tilde{g}_2(\tau)\, d\tau.$$

Evidently,

$$\frac{\partial G}{\partial \xi}(T - \tau, y, 0) = \frac{1}{4\sqrt{\pi}(T - \tau)^{3/2}} \sum_{n=-\infty}^{\infty} (y + 2nl) e^{-\frac{(y+2nl)^2}{4(T-\tau)}},\ 0 \le \tau \le T, 0 \le y \le l,$$

with singular term $\frac{y}{(T-\tau)^{3/2}} e^{-\frac{y^2}{4(T-\tau)}}$ for $\tau = T$, $y = 0$;

$$\frac{\partial G}{\partial \xi}(T - \tau, y, l) = \frac{1}{4\sqrt{\pi}(T - \tau)^{3/2}} \sum_{n=-\infty}^{\infty} (y + l(2n - 1)) e^{-\frac{(y+l(2n-1))^2}{4(T-\tau)}}$$

$$+ (y + l(2n + 1)) e^{-\frac{(y+l(2n+1))^2}{4(T-t)}},\ 0 \le t \le T, 0 \le y \le l,$$

with singular term $\frac{y-l}{(T-t)^{3/2}} e^{-\frac{(y-l)^2}{4(T-\tau)}}$ for $T = \tau$, $y = l$.

Put $a_n = 2nl$ or $a_n = (2n \pm 1)l$ and consider a single term in the previous series.

We shall prove at first the following lemma.

Lemma 11. *Let* $f(y) = \int_0^T \frac{y+a}{(T-\tau)^{3/2}} e^{-\frac{(y+a)^2}{4(T-\tau)}} h(\tau)\, d\tau$*, where* $h \in C([0,T])$ *and* a *is a real constant. Then* f *can be prolonged analytically in* $(y+i\theta)$ *in the open angle* $A = \{|y+a| > 2|\theta|,\ (y,\theta) \in \mathbf{R}^2\}$.

Certainly, A is the interior of two opposite angles with vertex at $(-a, 0)$.

Proof of Lemma 11. The change of the variable $\tau = T - \frac{1}{\gamma}$, i.e., $0 < \frac{1}{T-\tau} = \gamma$ leads to $f(y) = (y+a)\int_{\frac{1}{T}}^{\infty} \gamma^{-1/2} e^{-\frac{\gamma}{4}(y+a)^2} h(T - \frac{1}{\gamma})\, d\gamma$, i.e., $|f(y)| \le C|y + a| \int_{\frac{1}{T}}^{\infty} \gamma^{-1/2} e^{-\frac{\gamma}{4}(y+a)^2}\, d\gamma$; $C = \max_{0\le\tau\le T} |h(\tau)|$.

Certainly, $|y + a| > 0$.

Consider now

$$f(y+i\theta) = \int_{\frac{1}{T}}^{\infty} (y+a+i\theta)\gamma^{-1/2} e^{-\frac{\gamma}{4}(y+a+i\theta)^2} h\left(T - \frac{1}{\gamma}\right) d\gamma.$$

Evidently, $|e^{-\frac{\gamma}{4}(y+a+i\theta)^2}| = e^{-\frac{\gamma}{4}[(y+a)^2-\theta^2]}$. On the other hand, $(y+a)^2 - \theta^2 = (|y+a|+|\theta|)(|y+a|-|\theta|) > \frac{(y+a)^2}{2}$ in A and therefore $|e^{-\frac{\gamma}{4}(y+a+i\theta)^2}| \le e^{-\frac{\gamma}{4}(y+a)^2}$, while $|y+a| \le |y+a+i\theta| < \frac{3}{2}|y+a|$ in A. □

From geometrical reasons it is clear that $\cap_{n=-\infty}^{\infty}\{|y+2nl| > 2|\theta|\} \supset B$, B being the bounded parallelogram with vertices $(0,0)$, $(2l,0)$, $(l,\frac{l}{2})$, $(l,-\frac{l}{2})$, i.e., $B = \{0 < y < l, |y| > 2|\theta|\} \cup \{l < y < 2l, |y-2l| > 2|\theta|\}$.

In the case $a_n = (2n+1)l$ we have $\cap_{n=-\infty}^{\infty}\{|y+(2n+1)l| > 2|\theta|\} \supset C$, C being the bounded triangle $C = \{0 < y < l, |y-l| > 2|\theta|\}$.

Conclusion: Each term participating in the series $I(y)$, $II(y)$ can be prolonged analytically in the open parallelogram $\Delta = B \cap C = \{0 < y < l, y > 2|\theta|, l - y > 2|\theta|\} = \{|\theta| < \frac{y}{2}\} \cap \{|\theta| < \frac{y-l}{2}\}$.

We shall show now that both the series $I(y)$, $II(y)$ can be prolonged analytically in Δ. To do this we must find an uniform majorant in Δ of the series $S = \sum_{|n|\ge N}(y+a_n)\gamma^{-1/2}e^{-\frac{\gamma}{4}[(y+a_n)^2-\theta^2]}$, $N \gg 1$. As $\Gamma_0|n|^2 \ge |y+a_n|^2 \ge c_0|n|^2$, $c_0 = \text{const.} > 0$, $|n| \ge N$ we shall obtain in Δ the necessary majorant. Thus, $S \le C_0 \sum_{|n|\ge N} |n|\gamma^{-1/2}e^{-\frac{\gamma}{8}(y+a_n)^2} < C_0 \sum_{|n|\ge N} |n|\gamma^{-1/2}e^{-\alpha_0\gamma n^2}$ for some $C_0, \alpha_0 = \text{const.} > 0$.

The facts that $\gamma^{-1/2} \le T^{1/2}$, $n^2 \ge |n|$ and $e^{-\alpha_0\gamma|n|} = (e^{-\frac{\alpha_0}{2}\gamma|n|})^2$ enable us to obtain $S \le C_1\, T^{3/2} \sum_{|n|\ge N}(e^{-\frac{\alpha_0}{2}\gamma})^{|n|}, C_1 = \text{const.} > 0$.

In fact, $|n|e^{-\frac{\alpha_0}{2}\gamma|n|} \le |n|e^{-\frac{\alpha_0|n|}{2T}} \le \frac{2T}{\alpha_0}$.

The observation that for $\gamma \ge \frac{1}{T}$: $\sum_{|n|\ge 1}(e^{-\frac{\alpha_0\gamma}{2}})^{|n|} = \frac{e^{-\frac{\alpha_0\gamma}{2}}}{1-e^{-\frac{\alpha_0\gamma}{2}}} \le \frac{e^{-\frac{\alpha_0\gamma}{2}}}{1-e^{-\frac{\alpha_0}{2T}}} \in L_1(\gamma \ge \frac{1}{T})$ completes the proof of the analytic continuation of $I(y)$, $II(y)$. Going back to the coordinate $x_2 = y - \frac{l}{2}$, $0 < y < l$, applying (10), (11) and the analytic continuation of $III(y) = III(x_2 + \frac{l}{2})$ in the strip $\{|x_2| \le \frac{l}{2}\}$ we complete the proof of Theorem 5.

Proof of Theorem 7. As it is shown in [7], §10.6, the function

$$w_2(s,y) = -\int_0^l \int_0^s G(s,y;\tau,\xi)\tilde{\Phi}(\tau,\xi)\, d\xi d\tau$$

is a classical solution in ω_1 of the equation $Pw_2 = \tilde{\Phi}(s,y)$, $\Phi(t,x) = f^{\#}((mt)^{\frac{1}{m}}, x)$, $\tilde{\Phi}(0,0) = \tilde{\Phi}(0,1) = 0$.

Proof of the necessity. Combining (6), (7), (11) and $z(T,y) = 0$ we see that $w_1 = z - w_2$ satisfies the conditions of Theorem 7. The partial analytic hypoellipticity of P with respect to y leads us to the conclusion that $F_T(x_2 + l/2) = F_T(y)$ is an analytic function.

Proof of the sufficient condition. Consider now the function $z(s,y) = w_1(s,y) + w_2(s,y)$, where w_1 is defined in Theorem 7. Then $Pz = \tilde{\Phi}$, $z \in C^2(\omega_1) \cap C(\omega_1)$ and $z(s,y) = z(T - \frac{x_1^m}{m}, x_2 + \frac{l}{2})$, $0 \le x_1 \le (mT)^{\frac{1}{m}}$. Put

$$U(x_1,x_2) = \begin{cases} z(T - \frac{x_1^m}{m}, x_2 + \frac{l}{2}), x_1 \ge 0 \\ -z(T - \frac{x_1^m}{m}, x_2 + \frac{l}{2}), x_1 < 0 \end{cases}, \qquad (x_1,x_2) \in \omega.$$

The facts that $U(-x_1,x_2) = -U(x_1,x_2)$, $U(0,x_2) = 0$, $|x_2| \le \frac{l}{2}$ and

$$\frac{\partial U}{\partial x_1} = \begin{cases} -x_1^{2p+1} z_s'(T - \frac{x_1^m}{m}, x_2 + \frac{l}{2}) \\ x_1^{2p+1} z_s'(T - \frac{x_1^m}{m}, x_2 + \frac{l}{2}) \end{cases}$$

show that U is the solution of $Mu = x_1^{2p+1} f^{\#}$ we are looking for. $\square$

Remark 12. In fact, $F_T(x_2 + \frac{l}{2})$ is contained in a rather narrow class of analytic functions. To verify this consider the mixed problem $Pw_1 = 0$ in ω_1, $w_1|_{s=0} = \varphi(y)$, $0 \le y \le l$, $\varphi(0) = \varphi(l) = 0$, $\varphi \in C^1$, $w_1|_{y=0} = w_1|_{y=l} = 0$.

Then $w_1(T,y)$ can be prolonged as an odd function g of y on the torus $\mathbf{R}_y/2l\mathbf{Z}$ and $w_1(T,y) \in G_{1/2}(\mathbf{R}_y/2l\mathbf{Z})$, where $G_{1/2}$ stands for the corresponding Gevrey class of order 1/2. Conversely, assume that $g(y)$ is odd with respect to y, $g(y+2l) = g(y)$ and $g \in G_{1/2}(\mathbf{R}_y/2l\mathbf{Z})$. We suppose that the Fourier coefficients in the development $g(y) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi}{l} y$ satisfy the estimate $|B_n| \le C_n e^{-kn^2}$, $k = \text{const.} > 0$, $\{C_n\} \in l_1$. ($g \in G_{1/2} \Rightarrow |B_n| \le \text{const. } e^{-kn^2}$ but we impose a little bit stronger restriction on B_n.) Then if $k \ge T\pi^2/l^2$ there exist $w_1 \in C^0(\omega_1) \cap C^2(\omega_1)$ and $\varphi(y) \in C[0,l]$, $\varphi(0) = \varphi(l) = 0$ such that: $Pw_1 = 0$ in ω_1, $w_1|_{y=0} = w_1|_{y=l} = 0$, $w_1|_{s=0} = \varphi(y)$ and $w_1|_{s=T} = g(y)$. On the other hand, $g(y)$ is analytic in the above mentioned circle iff $|B_n| \le \alpha e^{-k|n|}$, $\alpha, k = \text{const.} > 0$. Thus: $\varphi \in C^1[0,l]$, $\varphi(0) = \varphi(l) = 0 \Rightarrow w_1(T,y) = g(y) \in G_{1/2}$. Conversely, assume that $g(y) \in G_{1/2-\varepsilon}$, $0 < \varepsilon < 1/2$, i.e., $g \in G_\alpha$, $0 < \alpha < \frac{1}{2}$, $\alpha = 1/2 - \varepsilon$ and with some $k > 0$ $|B_n| \le Ce^{-k|n|^{1/\alpha}} \Rightarrow \exists w_1 = \sum_1^{\infty} A_n e^{-\frac{n^2\pi^2}{l^2}s} \sin \frac{n\pi}{l} y$, $A_n = B_n e^{\frac{n^2\pi^2}{l^2}T}$, $|A_n| \le Ce^{|n|^{1/\alpha}(-k + n^{2-\frac{1}{\alpha}} \frac{\pi^2}{l^2} T)} \Rightarrow \varphi(y) = w_1(0,y) \in G_{1/2-\varepsilon}$, $w_1(s,0) = w_1(s,l) = 0$, $Pw_1 = 0$ in ω_1, $w_1(T,y) = g(y) \in G_{1/2-\varepsilon}$.

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The Solution of the Equation $\operatorname{div} \underline{w} = p \in L^2(\mathbb{R}^m)$ with $\underline{w} \in H_0^{1,2}(\mathbb{R}^m)^m$

Remigio Russo and Christian G. Simader

Abstract. The solution of the equation $\operatorname{div} \underline{w} = p$ is performed via suitable solutions of the Poisson equation. For this purpose appropriate Sobolev spaces and a certain non-standard negative norm have to be regarded.

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1. Introduction

While studying Stokes' system in a layer, we came to the following problem. Let $p \in L^2(\mathbb{R}^m)$ $(m \geq 1)$ and suppose that there is $\underline{w} \in \underline{H}^{1,2}(\mathbb{R}^m) := H^{1,2}(\mathbb{R}^m)^m$ $\left(\equiv H_0^{1,2}(\mathbb{R}^m)^m\right)$ $(H_0^{1,2}(\mathbb{R}^m) = W_0^{1,2}(\mathbb{R}^m)$ the classical Sobolev space) such that

$$p = \operatorname{div} \underline{w} := \sum_{i=1}^{m} \partial_i w_i. \tag{1.1}$$

Then for $\phi \in C_0^\infty(\mathbb{R}^m)$

$$\langle p, \phi \rangle = \langle \operatorname{div} \underline{w}, \phi \rangle = -\langle \underline{w}, \nabla \phi \rangle$$

and therefore

$$|\langle p, \phi \rangle| \leq \|\underline{w}\| \|\nabla \phi\| \quad \text{for } \phi \in C_0^\infty(\mathbb{R}^m). \tag{1.2}$$

Viceversa, if $p \in L^2(\mathbb{R}^m)$ is given, we want now to look for $\underline{w} \in \underline{H}^{1,2}(\mathbb{R}^m)$ so that (1.1) holds. A natural procedure would consist in solving the Poisson equation $-\Delta u = p$ with u in an appropriate function space and then to define $\underline{w} := -\nabla u$. For $\underline{w} \in \underline{H}^{1,2}(\mathbb{R}^m)$ to hold, necessarily u must satisfy $\partial_i u \in L^2(\mathbb{R}^m)$ and $\partial_i \partial_k u \in L^2(\mathbb{R}^m)$ for $i, k = 1, \ldots, m$. This gives a first hint for the appropriate choice of the function space where we have to look for solutions. In addition, the set of admissible data, because of (1.2), is restricted to those $p \in L^2(\mathbb{R}^m)$ satisfying

$$\sup_{0 \neq \phi \in C_0^\infty(\mathbb{R}^m)} \frac{\langle p, \phi \rangle}{\|\nabla \phi\|} < \infty.$$

In the sequel we first study systematically the space where we are looking for solutions u of the Poisson equation $-\Delta u = p$, then for the space of data p. Finally we are able to solve that Poisson equation and we get a solution being as well a weak as a strong solution of that equation.

2. The appropriate function space for solutions

Let $B := \{x \in \mathbb{R}^m : |x| < 1\}$. Then we put

$$\begin{aligned} D_B^2 \equiv D_B^2(\mathbb{R}^m) := \{u \in L^2_{\text{loc}}(\mathbb{R}^m) :\ & \text{There exist weakly } \partial_i u \in L^2(\mathbb{R}^m) \\ & \text{and } \partial_i \partial_j u \in L^2(\mathbb{R}^m) \text{ for } i, j = 1, \dots, m \text{ and } \int_B u dy = 0\}. \end{aligned} \tag{2.1}$$

The condition $\int_B u dy = 0$ rules out the constants $u \equiv c \neq 0$. Clearly instead of B we could choose any other $\emptyset \neq G \subset\subset \mathbb{R}^m$. Then the resulting spaces are isomorphic one to the other (and even isometrically isomorphic with the norm introduced below in (2.2)). Let us emphasize that $u \in D_B^2(\mathbb{R}^m)$ *does not* necessarily imply $u \in H^{2,2}(\mathbb{R}^m)$. Let, e.g., $m \geq 3$ and let $\alpha \in \mathbb{R}$ satisfy $-\frac{m}{2} < \alpha < 1 - \frac{m}{2}$. It is readily seen that

$$u(x) := |x|^\alpha (1 - \eta_2(x)), \qquad x \in \mathbb{R}^m$$

(where η_2 is defined by (2.8)) satisfies $u \in D_B^2(\mathbb{R}^m)$, but $u \notin L^2(\mathbb{R}^m)$, whence $u \notin H^{2,2}(\mathbb{R}^m)$. Furthermore, with $p := -\Delta u$ we see $p \in L^2(\mathbb{R}^m)$ and $p \in L^{\frac{2m}{m+2}}(\mathbb{R}^m)$, therefore by (3.3) $p \in L^2_\Delta(\mathbb{R}^m)$ (compare (3.1)). For $u, v \in D_B^2$ we set (for $f, g \in L^2(\mathbb{R}^m)$ let $\langle f, g \rangle := \int_{\mathbb{R}^m} f g dx$)

$$[u, v]_2 := \sum_{i,j=1}^m \langle \partial_i \partial_j u, \partial_i \partial_j v \rangle$$

and

$$\langle u, v \rangle_2 := \langle \nabla u, \nabla v \rangle + [u, v]_2. \tag{2.2}$$

In the sequel we often use Poincaré's inequality and a slight extension of it. For $R > 0$ let

$$\begin{cases} A_R & := \{x \in \mathbb{R}^m : R < |x| < 2R\} \\ B_R & := \{x \in \mathbb{R}^m : |x| < R\} \quad B := B_1 \text{ for } R = 1 \end{cases} \tag{2.3}$$

and let Ω_R denote either A_R or B_R. Then with a constant $C_{POI} = C(\Omega_1) > 0$

$$\|v\|_{\Omega_R} \leq C_{POI} R \|\nabla v\|_{\Omega_R} \qquad \forall v \in H^{1,2}(\Omega_R) \text{ with } \int_{\Omega_R} v dy = 0. \tag{2.4}$$

In case $\Omega_R = B_R$ and $R \geq 1$ it is readily seen by means of the Schwarz inequality

$$\|u\|_{B_R} \leq C(R)\|\nabla u\|_{B_R} \qquad \forall u \in H^{1,2}(B_R) \text{ with } \int_B u dy = 0, \tag{2.5}$$

where

$$C(R) = RC_{POI}\left[1 - (1 - R^{-m})^{\frac{1}{2}}\right]^{-1}.$$

From the definition (2.1) it follows immediately for $1 \leq R < \infty$ that $u \mid_{B_R} \in H^{2,2}(B_R)$ if $u \in D_B^2$. If $u \in D_B^2$ and $\langle u, u\rangle_2 = 0$, then $\|\nabla u\|_{2,\mathbb{R}^m} = 0$ and because of (2.5) $\|u\|_{2,B_R} = 0$ for every $1 \leq R < \infty$, whence $u = 0$ a.e. in $\mathbb{R}^m$. Therefore, by (2.2) an inner product is defined on D_B^2 (all other properties are obvious). We put

$$\|u\|_2 := \sqrt{\langle u, u\rangle_2} \quad \text{for} \quad u \in D_B^2. \tag{2.6}$$

Theorem 2.1. D_B^2 *equipped with the inner product defined by* (2.2) *is a Hilbert space.*

Proof. It remains to prove completeness. If $(u_\nu) \subset D_B^2$ is Cauchy, then

$$\|\partial_i u_\nu - \partial_i u_\nu\| \to 0 \text{ and } \|\partial_i\partial_k u_\nu - \partial_i\partial_k u_\mu\| \to 0$$

as $\nu, \mu \to \infty$ for $i, k = 1, \dots, m$. Because of completeness of $L^2(\mathbb{R}^m)$ there are $v_i, v_{ik} \in L^2(\mathbb{R}^m)$ so that $\|v_i - \partial_i u_\nu\| \to 0$ and $\|v_{ik} - \partial_i\partial_k u_\nu\| \to 0$ as $\nu \to \infty$ for $i, k = 1, \dots, m$. For every $n \in \mathbb{N}$ because of (2.5) we see

$$\|u_\nu - u_\mu\|_{2,B_n} \leq C(n)\|\nabla u_\nu - \nabla u_\mu\|_{2,B_n} \to 0$$

as $\nu, \mu \to \infty$. Then there is $u^{(n)} \in L^2(B_n)$ so that $\|u^{(n)} - u_\nu\|_{2,B_n} \to 0 (\nu \to \infty)$. Clearly $u^{(n+1)} \mid_{B_n} = u^{(n)}$ a.e. in B_n. Beginning with $n = 1$ we may successively choose suitable representatives in every equivalence class $u^{(n+1)}$, so that $u^{(n+1)}(x) = u^{(n)}(x)$ for all $x \in B_n$. Then a measurable function $u : \mathbb{R}^m \to \mathbb{R}$ is defined by $u(x) := u^{(n)}(x)$ for $x \in B_n$. Clearly $u \in L^2_{\text{loc}}(\mathbb{R}^m)$ by construction and $\int_B u dy = \lim_{\nu\to\infty} \int_B u_\nu dy = 0$. If $\varphi \in C_0^\infty(\mathbb{R}^m)$ there is $n_0 = n(\varphi) \in \mathbb{N}$ so that $\operatorname{supp}\varphi \subset B_{n_0}$. Then for $i = 1, \dots, m$

$$\int_{\mathbb{R}^m} u\partial_i\varphi dx = \int_{B_{n_0}} u\partial_i\varphi dx = \lim_{\nu\to\infty} \int_{B_{n_0}} u_\nu\partial_i\varphi dx = -\lim_{\nu\to\infty} \int_{B_{n_0}} \partial_i u_\nu\varphi dx = -\int_{\mathbb{R}^n} v_i\varphi dx$$

Therefore v_i is the weak ∂_i-derivative of u. Analogously we see $\partial_i\partial_k u = v_{ik} \in L^2(\mathbb{R}^m)$. But then $u \in D_B^2$. $\square$

For an intermediate step (existence of weak solutions to Poisson's equation) we need additional Sobolev spaces. For detailed further information we refer to [1], Section 2 (there the spaces are denoted by $\hat{H}^{1,2}(\mathbb{R}^m)$, etc.). We put

$$L^{1,2}(\mathbb{R}^m) := \left\{u \in L^2_{\text{loc}}(\mathbb{R}^m) : \text{ There exist weakly } \partial_i u \in L^2(\mathbb{R}^m),\ i = 1, \dots, m\right\}.$$

We want now to construct suitable subspaces of $L^{1,2}(\mathbb{R}^m)$ (depending whether $m > 2$ or $m \leq 2$) being isometrically isomorphic to the (abstract Cantor-) completion of $C_0^\infty(\mathbb{R}^m)$ with respect to $\|\nabla.\|$–norm. For more details see [1], Section 2. Let first $m > 2$. Then Sobolev's inequality

$$\|\phi\|_{2^*} \le C_{SOB}\|\nabla\phi\| \quad \text{for } \phi \in C_0^\infty(\mathbb{R}^m) \tag{2.7}$$

holds with $C_{SOB} = C(m) > 0$ and $2^* := \frac{2m}{m-2}$.

Theorem 2.2. *Let $m \ge 3$ and let*

$$\hat{H}_0^{1,2}(\mathbb{R}^m) := \left\{ u \in L^{1,2}(\mathbb{R}^m) : u \in L^{2^*}(\mathbb{R}^m) \right\}.$$

Then

1. *for $u, v \in \hat{H}_0^{1,2}(\mathbb{R}^m)$ by $\langle \nabla u, \nabla v\rangle$ an inner product is defined on $\hat{H}_0^{1,2}(\mathbb{R}^m)$ such that $\hat{H}_0^{1,2}(\mathbb{R}^m)$ equipped with this inner product is a Hilbert space.*
2. *$C_0^\infty(\mathbb{R}^m)$ is dense in $\hat{H}_0^{1,2}(\mathbb{R}^m)$ with respect to $\|\nabla.\|$.*

Proof. For $R = k \in \mathbb{N}$ let $\eta_k \in C_0^\infty(\mathbb{R}^m)$ be defined by

$$\begin{cases} \eta \in C_0^\infty(\mathbb{R}^m), \quad 0 \le \eta \le 1, \\ \eta(x) = 1 \text{ for } |x| \le 1 \text{ and } \eta(x) = 0 \text{ for } |x| \ge 2, \\ \eta_R(x) := \eta\left(\frac{x}{R}\right) \text{ for } R > 0. \end{cases} \tag{2.8}$$

If $R = k \in \mathbb{N}$, then with A_k by (2.3) we see

$$|\nabla\eta_k(x)| \le Ck^{-1}\chi_{A_k}(x), \quad |\partial_i\partial_j\eta_k(x)| \le Ck^{-2}\chi_{A_k}(x). \tag{2.9}$$

For $u \in \hat{H}_0^{1,2}(\mathbb{R}^m)$ we see $\eta_k \cdot u \in L^{2^*}$ and $\|u - \eta_k u\|_{2^*} = \|(1-\eta_k)u\|_{2^*} \to 0$ $(k \to \infty)$ by Lebesgue's theorem. Further,

$$\nabla(\eta_k u) = \eta_k \nabla u + \nabla\eta_k u.$$

Again by Lebesgue's theorem, $\|\nabla u - \eta_k\nabla u\| \to 0$. Because of (2.8) and Hölder's inequality

$$\|(\nabla\eta_k)u\|^2 \le C^2k^{-2}\int_{A_k}|u|^2dx \le C^2k^{-2}\left(\int_{A_k}|u|^{\frac{2m}{m-2}}dx\right)^{\frac{m-2}{m}} \cdot |A_k|^{\frac{2}{m}}.$$

Since $k^{-2}|A_k|^{\frac{2}{m}} = \text{const}(m)$ for all $k \in \mathbb{N}$ and $\int_{A_k}|u|^{\frac{2m}{m-2}dx} \to 0$ $(k \to \infty)$, we see $\|(\nabla\eta_k)u\| \to 0$ $(k \to \infty)$, whence $\|\nabla u - \nabla(\eta_k u)\| \to 0$ $(k \to \infty)$. For $\varepsilon > 0$ consider the mollifications $(\eta_k u)_\varepsilon$. Since $\nabla(\eta_k u)_\varepsilon(x) = (\nabla(\eta_k u))_\varepsilon(x)$ for $\varepsilon > 0$ and $x \in \mathbb{R}^m$ we see

$$\|\nabla(\eta_k u) - \nabla(\eta_k u)_\varepsilon\| \to 0, \qquad \|\eta_k u - (\eta_k u)_\varepsilon\|_{2^*} \to 0 \ (\varepsilon \to 0).$$

For $k \in \mathbb{N}$ there exists $\varepsilon_k > 0$ such that with $v_k := (\eta_k u)_{\varepsilon_k} \in C_0^\infty(\mathbb{R}^m)$ it holds $\|\nabla(\eta_k u) - \nabla v_k\| \le \frac{1}{k}$ and $\|\eta_k u - v_k\|_{2^*} \le \frac{1}{k}$, therefore

$$\|\nabla u - \nabla v_k\| + \|u - v_k\|_{2^*} \to 0 \qquad (k \to \infty).$$

For v_k $(k \in \mathbb{N})$ Sobolev's inequality (2.7) holds true and therefore finally for $k \to \infty$

$$\|u\|_{2^*} \le C_{SOB}\|\nabla u\| \qquad \forall u \in \hat{H}_0^{1,2}(\mathbb{R}^n). \tag{2.10}$$

Because of (2.10) $\|\nabla.\|$ is a norm on $\hat{H}_0^{1,2}(\mathbb{R}^n)$. Furthermore, by part 1 of the proof, $C_0^\infty(\mathbb{R}^m)$ is dense in $\hat{H}_0^{1,2}(\mathbb{R}^m)$ with respect to that norm. To prove completeness, let now $(u_k) \subset \hat{H}_0^{1,2}(\mathbb{R}^m)$ be Cauchy with respect to $\|\nabla.\|$. Because of (2.10) and completeness of $L^{2^*}(\mathbb{R}^m)$ respectively $L^2(\mathbb{R}^m)^m$ there exist $u \in L^{2^*}(\mathbb{R}^m)$ and $f \in L^2(\mathbb{R}^m)^m$ such that

$$\|u - u_k\|_{2^*} \to 0, \quad \|f - \nabla u_k\|_{L^2(\mathbb{R}^m)^m} \to 0 \quad (k \to \infty).$$

For $\phi \in C_0^\infty(\mathbb{R}^m)$

$$\int\limits_{\mathbb{R}^m} u\partial_i\phi = \lim_{k\to\infty} \int\limits_{\mathbb{R}^m} u_k\partial_i\phi = -\lim_{k\to\infty} \int\limits_{\mathbb{R}^m} \partial_i u_k\phi = -\int\limits_{\mathbb{R}^m} f_i\phi,$$

hence $\partial_i u = f_i \in L^2(\mathbb{R}^m)$. Therefore $u \in \hat{H}_0^{1,2}(\mathbb{R}^m)$ and $\|\nabla u - \nabla u_k\| \to 0$ $(k \to \infty)$. □

For $m \in \mathbb{N}$ we consider with $B = B_1$

$$L_B^{1,2}(\mathbb{R}^m) := \{u \in L^{1,2}(\mathbb{R}^m) : \int\limits_B u dy = 0\}. \tag{2.11}$$

Theorem 2.3. *For $u, v \in L_B^{1,2}(\mathbb{R}^m)$ by $\langle\nabla u, \nabla v\rangle$ an inner product is defined on $L_B^{1,2}(\mathbb{R}^m)$ such that $L_B^{1,2}(\mathbb{R}^m)$ is a Hilbert space. For every $u \in L_B^{1,2}(\mathbb{R}^m)$ there is a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^m)$ such that (in the sense of a seminorm) $\|\nabla u - \nabla u_n\| \to 0$.*

Proof. If $u \in L_B^{1,2}(\mathbb{R}^m)$ and $\|\nabla u\| = 0$, then by (2.5) for arbitrary $n \in \mathbb{N}$

$$\|u\|_{B_n} \leq C(n)\|\nabla u\|_{B_n} = 0.$$

Hence $u \mid_{B_n} = 0$ a.e. for all $n \in \mathbb{N}$ and therefore $u = 0$ a.e. in $\mathbb{R}^m$. All other properties of an inner product are obvious. The proof of completeness of $L_B^{1,2}(\mathbb{R}^m)$ is performed by means of (2.5) completely analogous to the proof of Theorem 2.1. Let

$$d_n := \frac{1}{|A_n|} \int\limits_{A_n} u(y)dy.$$

We put $v_n := \eta_n(u - d_n)$. Then

$$\partial_i v_n = \partial_i\eta_n(u - d_n) + \eta_n\partial_i u.$$

By Lebesgue's theorem, $\|(1 - \eta_n)\partial_i u\| \to 0$ $(n \to \infty)$. Because of (2.9) we see with the help of (2.4)

$$\|\partial_i\eta_n(u - c_n)\| \leq Cn^{-1}\|u - c_n\|_{2,A_n} \leq C \cdot C_{POI}\|\nabla u\|_{2,A_n} \to 0$$

as $n \to \infty$. Therefore $\|\nabla u - \nabla v_n\| \to 0$. Clearly, by (2.9) $\|\partial_i\eta_n\partial_j u\| \to 0$ $(n \to \infty)$. We use a standard mollifier kernel and we find for each fixed $n \in \mathbb{N}$ an $0 < \varepsilon_n < 1$ so that

$$\|\nabla v_n - \nabla(v_n)_{\varepsilon_n}\| \leq n^{-1}.$$

Here we made use of the fact that in $\mathbb{R}^m$ $\partial_i v_\varepsilon = (\partial_i v)_\varepsilon$. Then $u_n := (v_n)_{\varepsilon_n} \in C_0^\infty(\mathbb{R}^m)$ (even $\operatorname{supp} u_n \subset B_{2n+1}$) and in the sense of a *seminorm* $\|\nabla u - \nabla u_n\| \to 0$ as $n \to \infty$. □

Lemma 2.4. *With η by (2.11) and for $3 \le k \in \mathbb{N}$ let*

$$\sigma_k(x) := \begin{cases} 1 & \text{for} \quad |x| \le e^e \\ \eta\left(\frac{\ln\ln|x|}{\ln\ln k}\right) & \text{for} \quad |x| \ge e^e \end{cases} \tag{2.12}$$

Then, $\sigma_k \in C_0^\infty(\mathbb{R}^m)$ $(m \ge 1)$, $\sigma_k(x) = 1$ for $|x| \le k$ and $\sigma_k(x) = 0$ for $|x| \ge e^{(\ln k)^2}$. In addition

$$\|\nabla\sigma_k\| \to 0 \quad (k \to \infty) \text{ for } m = 1, 2. \tag{2.13}$$

Proof. Let $R_k := \left\{x \in \mathbb{R}^m : k < |x| < e^{(\ln k)^2}\right\}$. Then

$$|\nabla\sigma_k(x)| \le \|\nabla\eta\|_\infty (\ln\ln k)^{-1}(|x|\ln|x|)^{-1}\chi_{R_k}(x)$$

in case $m = 2$ using polar coordinates,

$$\int_{R_k} \frac{dx}{|x|^2(\ln|x|)^2} = 2\pi \int_k^{e^{(\ln k)^2}} \frac{dr}{r(\ln r)^2} = -2\pi(\ln r)^{-1}\Big|_k^{e^{(\ln k)^2}} \le 2\pi\frac{1}{\ln k} \to 0.$$

In case $m = 1$

$$\int_{R_k} \frac{dx}{x^2(\ln|x|)^2} = 2\int_k^{e^{(\ln k)^2}} \frac{dx}{x^2(\ln x)^2} \le \frac{2}{k}\int_k^{e^{(\ln k)^2}} \frac{dx}{x(\ln x)^2} \le \frac{2}{k}\,\frac{1}{\ln k} \to 0.$$

□

Lemma 2.5. *Let $m = 1$ or 2 and let $(B := B_1)$*

$$C_{0;B}^\infty(\mathbb{R}^m) := \left\{\phi \in C_0^\infty(\mathbb{R}^m) : \int_B \phi(y)dy = 0\right\}.$$

Then, for every $\phi \in C_0^\infty(\mathbb{R}^m)$ there is a sequence $(\phi_k) \subset C_{0;B}^\infty(\mathbb{R}^m)$ such that

$$\|\nabla\phi - \nabla\phi_k\| \to 0.$$

Proof. Let $c_\phi := \frac{1}{|B|}\int_B \phi(y)dy$ and let $\phi_k := \phi - c_\phi\sigma_k$ for $k \ge 3$. Since $\sigma_k\,|_B = 1$, we see $\int_B \phi_k dx = 0$. By (2.13),

$$\|\nabla\phi - \nabla\phi_k\| = |c_\phi|\|\nabla\sigma_k\| \to 0.$$

□

Corollary 2.6. *Let $m = 1$ or 2. Then $C_{0;B}^\infty(\mathbb{R}^m)$ is a dense subspace of $L_B^{1,2}(\mathbb{R}^m)$.*

Proof. Given $u \in L_B^{1,2}(\mathbb{R}^m)$, by Theorem 2.3 there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^m)$ such that $\|\nabla u - \nabla u_n\| \to 0$. By Lemma 2.5, for every $n \in \mathbb{N}$ there exists $v_n \in C_{0;B}^\infty(\mathbb{R}^m)$ such that $\|\nabla u_n - \nabla v_n\| \le \frac{1}{n}$. Hence $\|\nabla u - \nabla v_n\| \to 0$. □

3. The space of data

By (1.2) it is suggested to consider

$$L^2_\Delta \equiv L^2_\Delta(\mathbb{R}^m) := \left\{ p \in L^2(\mathbb{R}^m) : \sup_{0\neq\phi\in C_0^\infty(\mathbb{R}^m)} \frac{\langle p,\phi\rangle}{\|\nabla\phi\|} < \infty \right\}. \tag{3.1}$$

We set

$$|p|_{-1} := \sup_{0\neq\phi\in C_0^\infty(\mathbb{R}^m)} \frac{\langle p,\phi\rangle}{\|\nabla\phi\|} \text{ for } p \in L^2_\Delta, \tag{3.2}$$

$$\|p\|_{-1} := \left(\|p\|^2 + |p|^2_{-1}\right)^{\frac{1}{2}}.$$

It is readily seen that $|.|_{-1}$ is even a norm on L^2_Δ. In fact, if $p \in L^2_\Delta$ and $|p|_{-1} = 0$, then with a usual mollifier kernel ($0 \le j \in C_0^\infty(\mathbb{R}^m)$, $j(-x) = j(x)$, $j(x) = 0$ for $|x| \ge 1$ and $\int\limits_{\mathbb{R}^m} j(x)dx = 1$; for $\varepsilon > 0$ let $j_\varepsilon(x) := \varepsilon^{-m} j\left(\frac{x}{\varepsilon}\right)$) we put $\phi(y) := j_\varepsilon(x-y)$. Then $0 = \langle p,\phi\rangle = \int p(y)j_\varepsilon(x-y)dy = p_\varepsilon(x)$. Since $\|p - p_\varepsilon\| \to 0$ for $\varepsilon \to 0$ we see $p = 0$ a.e. But L^2_Δ equipped with this norm needs not to be complete. On the contrary, L^2_Δ equipped with the norm $\|.\|_{-1}$ is a Banach space (see Corollary 4.3 below). In case $m = 1, 2$ we observe that with σ_k by (2.12) for $p \in L^2_\Delta(\mathbb{R}^m)$ because of (2.13)

$$|\langle p, \sigma_k\rangle| \le |p|_{-1}\|\nabla\sigma_k\| \to 0 \qquad (k \to \infty).$$

In [2] we studied several *sufficient* conditions for $p \in L^2_\Delta(\mathbb{R}^m)$. E.g., if $m \ge 3$ and $p \in L^{\frac{2m}{m+2}}(\mathbb{R}^m)$ then by (2.7) we see

$$|p|_{-1} \le C_{SOB}\|p\|_{L^{\frac{2m}{m+2}}(\mathbb{R}^m)}. \tag{3.3}$$

The case $m = 1, 2$ can be treated via Hardy inequalities. Clearly, with the usual norm (for $p \in L^2(\mathbb{R}^m)$)

$$\|p\|_{H^{-1,2}(\mathbb{R}^m)} := \sup\left\{ \frac{\langle p,\phi\rangle}{(\|\phi\|^2 + \|\nabla\phi\|^2)^{\frac{1}{2}}} : 0 \neq \phi \in C_0^\infty(\mathbb{R}^m) \right\}$$

we see immediately

$$\|p\|_{H^{-1,2}(\mathbb{R}^m)} \le |p|_{-1}.$$

If $m = 1, 2$ and even for $p \in C_0^\infty(\mathbb{R}^m)$ it needs not to hold $|p|_{-1} < \infty$. Suppose $|p|_{-1} < \infty$. Then for $p, \phi \in C_0^\infty(\mathbb{R}^m)$ denote by $\hat{p}$, $\hat{\phi} \in S(\mathbb{R}^m)$ the Fourier transform. Then

$$|\langle p,\phi\rangle| = \left| \int\limits_{\mathbb{R}^m} \hat{p}^*(\xi)\hat{\phi}(\xi)d\xi \right| \le |p|_{-1} \left(\int\limits_{\mathbb{R}^m} |\hat{\phi}(\xi)|^2|\xi|^2 \right)^{\frac{1}{2}}$$

is valid first for $\phi \in C_0^\infty(\mathbb{R}^m)$ and then even for $\phi \in S(\mathbb{R}^m)$ (= Schwartz class). Let $\eta > 0$ and let $\hat{\phi}(\xi) := \frac{\hat{p}(\xi)}{\eta+|\xi|^2}$. Then $\hat{\phi} \in S(\mathbb{R}^m)$, hence $\phi := \tilde{\hat{\phi}} \in S(\mathbb{R}^n)$ (where

$\tilde{f}$ denotes the inverse Fourier transform). Then we conclude

$$\int\limits_{\mathbb{R}^m} \frac{|\hat{p}(\xi)|^2}{\eta + |\xi|^2} d\xi \leq |p|_{-1} \left(\int\limits_{\mathbb{R}^m} \frac{|\hat{p}(\xi)|^2 |\xi|^2}{(\eta + |\xi|^2)^2} d\xi \right)^{\frac{1}{2}} \leq |p|_{-1} \left(\int\limits_{\mathbb{R}^m} \frac{|\hat{p}(\xi)|^2}{\eta + |\xi|^2} d\xi \right)^{\frac{1}{2}} .$$

For $\eta \to 0$ by Fatou's lemma

$$\int\limits_{\mathbb{R}^m} \frac{|\hat{p}(\xi)|^2}{|\xi|^2} d\xi \leq |p|_{-1}^2. \tag{3.4}$$

As an example, let for $m = 1$

$$p(x) := \begin{cases} \frac{2}{\sqrt{2\pi}} \frac{\sin x}{x}, & x \neq 0 \\ \frac{2}{\sqrt{2\pi}}, & x = 0. \end{cases}$$

Then $p \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and

$$\hat{p}(\xi) = \begin{cases} 1 & \text{for } |x| \leq 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

and the left-hand side of (3.4) is *not* finite, hence $|p|_{-1} = \infty$.

4. Solution of $-\Delta u = p$ for $p \in L^2_\Delta(\mathbb{R}^m)$.

Lemma 4.1. *For $p \in L^2_\Delta(\mathbb{R}^m)$ there is a unique $u \in L^{1,2}_B(\mathbb{R}^m)$ so that*

$$\langle \nabla u, \nabla \phi \rangle = \langle p, \phi \rangle \qquad \forall \phi \in C_0^\infty(\mathbb{R}^m) \tag{4.1}$$

and

$$\|\nabla u\| = |p|_{-1}. \tag{4.2}$$

Proof. To prove existence we set

$$H := \begin{cases} \hat{H}_0^{1,2}(\mathbb{R}^m) & \text{if } m \geq 3 \\ L_B^{1,2}(\mathbb{R}^m) & \text{if } m = 1, 2, \end{cases}$$

$$C := \begin{cases} C_0^\infty(\mathbb{R}^m) & \text{if } m \geq 3 \\ C_{0;B}^\infty(\mathbb{R}^m) & \text{if } m = 1, 2. \end{cases}$$

Then C is dense in the Hilbert space H with respect to $\|\nabla.\|$-norm ($m \geq 3$: Theorem 2.2, $m = 1, 2$: Corollary 2.6). Let

$$F^*(\phi) := \langle p, \phi \rangle \text{ for } \phi \in C.$$

Then

$$|F^*(\phi)| \leq |p|_{-1} \|\nabla \phi\| \text{ for } \phi \in C.$$

Therefore, F^* is a densely defined continuous linear functional having a unique norm-preserving extension $\tilde{F}^* \in H^*$. By the Riesz-Fréchet representation theorem there is a unique $u \in H$ such that

$$\langle \nabla u, \nabla \phi \rangle = \tilde{F}^*(\phi) = \langle p, \phi \rangle \qquad \forall \phi \in C. \tag{4.3}$$

and

$$\|\nabla u\| = \sup \left\{ \frac{\langle p, \phi \rangle}{\|\nabla \phi\|} : 0 \neq \phi \in C \right\} = |p|_{-1}. \tag{4.4}$$

If $m = 1, 2$ and $\phi \in C_0^\infty(\mathbb{R}^m)$, we regard

$$\begin{cases} c_\phi & := \frac{1}{|B|} \int\limits_B \phi(y)dy \quad \text{ and} \\ \varphi_k & := \phi - c_\phi \sigma_k \in C_{0;B}^\infty(\mathbb{R}^m) \quad \text{ for } m = 1, 2. \end{cases}$$

Since

$$|\langle \nabla u, \nabla \sigma_k \rangle| \leq \|\nabla u\| \, \|\nabla \sigma_k\| \to 0,$$
$$|\langle p, \nabla \sigma_k \rangle| \leq |p|_{-1} \|\nabla \sigma_k\| \to 0$$

we see $\langle \nabla u, \nabla \varphi_k \rangle \to \langle \nabla u, \nabla \phi \rangle$, $\langle p, \varphi_k \rangle \to \langle p, \phi \rangle$. Since (4.3) holds for all φ_k, it follows

$$\langle \nabla u, \nabla \phi \rangle = \langle p, \phi \rangle \qquad \forall \phi \in C_0^\infty(\mathbb{R}^m) \qquad \forall m \geq 1.$$

On the other hand,

$$|p|_{-1} = \sup_{0 \neq \phi \in C_0^\infty(\mathbb{R}^m)} \frac{\langle p, \phi \rangle}{\|\nabla \phi\|} = \sup_{0 \neq \phi \in C_0^\infty(\mathbb{R}^m)} \frac{\langle \nabla u, \nabla \phi \rangle}{\|\nabla \phi\|} \leq \|\nabla u\| \tag{4.5}$$

and it follows (4.2) from (4.4). In case $m \geq 3$ we replace u by $(u - c_u) \in L_B^{1,2}(\mathbb{R}^m)$, where $c_u := \frac{1}{|B|} \int\limits_B u(y)dy$, $\nabla(u - c_u) = \nabla u$. Suppose now that for $m \geq 1$ there is a further solution $v \in L_B^{1,2}(\mathbb{R}^m)$ of (4.1). Then $h := u - v \in L_B^{1,2}(\mathbb{R}^m)$ and

$$\langle \nabla h, \nabla \phi \rangle = 0 \qquad \forall \phi \in C_0^\infty(\mathbb{R}^m).$$

By Theorem 2.3 there is a sequence $(h_k) \subset C_0^\infty(\mathbb{R}^m)$ such that $\|\nabla h - \nabla h_k\| \to 0$. Then

$$\|\nabla h\|^2 = \langle \nabla h, \nabla h \rangle - \langle \nabla h, \nabla h_k \rangle \leq \|\nabla h\| \, \|\nabla h - \nabla h_k\| \to 0,$$

whence $h = 0$. $\square$

Completely analogous to part 3 of the proof of Theorem 2.3 (consider again for $u \in D_B^2$ the functions $v_n := \eta_n(u - d_n)$ and mollify!) it can be proved easily that for $u \in D_B^2$ given, there exists a sequence $(u_n) \subset C_0^\infty(\mathbb{R}^m)$ such that $\|u - u_n\|_2 \to 0$ $(n \to \infty)$ in the sense of a seminorm. Hence, integrating twice by parts,

$$[u, v]_2 = \lim_{n \to \infty} \sum_{i,j=1}^m \langle \partial_i \partial_j u_n, \partial_i \partial_j v \rangle = \lim_{n \to \infty} \langle \Delta u_n, \Delta v \rangle = \langle \Delta u, \Delta v \rangle$$

Therefore

$$\|u\|_2 = \left(\|\nabla u\|^2 + \|\Delta u\|^2 \right)^{\frac{1}{2}} \quad \text{ for } u \in D_B^2. \tag{4.6}$$

Theorem 4.2. *If $u \in D^2_B(\mathbb{R}^m)$, then $-\Delta u =: p \in L^2_\Delta(\mathbb{R}^m)$. Furthermore, the map*

$$-\Delta : D^2_B(\mathbb{R}^m) \to L^2_\Delta(\mathbb{R}^m)$$
$$u \to -\Delta u$$

is an isometric isomorphism (*compare* (2.2), (2.6), (4.6), (3.3))*:*

$$\|\Delta u\|_{-1} = \|u\|_2 = \|p\|_{-1} \text{ and } \|\nabla u\| = |p|_{-1} \qquad \forall u \in D^2_B(\mathbb{R}^m). \tag{4.7}$$

Proof. Let $u \in D^2_B(\mathbb{R}^m) \subset L^{1,2}_B(\mathbb{R}^m)$ and put $p := -\Delta u$. Then for $\phi \in C_0^\infty(\mathbb{R}^m)$ $\langle p, \phi\rangle = \langle \nabla u, \nabla\phi\rangle$. Then $|\langle p,\phi\rangle| \le \|\nabla u\|\ \|\nabla \phi\|\ \forall \phi \in C_0^\infty(\mathbb{R}^m)$, therefore $|p|_{-1} \le \|\nabla u\|$. By Theorem 2.3 there exists a sequence $(u_k) \subset C_0^\infty(\mathbb{R}^m)$ such that $\|\nabla u - \nabla u_k\| \to 0$. Then

$$\|\nabla u\|^2 = \lim_{k\to\infty} |\langle \nabla u, \nabla u_k\rangle| \le \lim_{k\to\infty} |p|_{-1}\|\nabla u_k\| = |p|_{-1}\|\nabla u\|.$$

Therefore $\|\nabla u\| = |p|_{-1}$. Then, by (3.3), (2.6), (4.5) (since clearly $p \in L^2(\mathbb{R}^m)$)

$$\|p\|^2_{-1} = \|p\|^2 + |p|^2_{-1} = \|\Delta u\|^2 + \|\nabla u\|^2 = \|u\|_2^2$$

proving (4.7). Clearly, $-\Delta$ is linear and because of (4.7) injective. To prove surjectivity, let $p \in L^2_\Delta(\mathbb{R}^m)$ be given. By Lemma 4.1 there exists a unique $u \in L^{1,2}_B(\mathbb{R}^m)$ such that (4.1) holds. We use again a standard mollifier kernel and for $\phi \in C_0^\infty(\mathbb{R}^m)$ we put $\phi_\varepsilon \in C_0^\infty(\mathbb{R}^m)$ in (4.1). Using the fact that the mollifier is a Hermitian operator that commutes on $\mathbb{R}^m$ with differentiation we get $\langle \nabla u_\varepsilon, \nabla\phi\rangle = \langle p_\varepsilon, \phi\rangle$ and therefore $\langle -\Delta u_\varepsilon - p_\varepsilon, \phi\rangle = 0$ for all $\phi \in C_0^\infty(\mathbb{R}^m)$. Since $u_\varepsilon \in C^\infty(\mathbb{R}^m) \subset L^2_{\text{loc}}(\mathbb{R}^m)$ we conclude $-\Delta u_\varepsilon = p_\varepsilon$. With $c_\varepsilon := \frac{1}{|B|}\int\limits_B u_\varepsilon(y)dy$ we see that $\tilde u_\varepsilon := (u_\varepsilon - c_\varepsilon) \in L^{1,2}_B(\mathbb{R}^m)$ and

$$\langle \nabla \tilde u_\varepsilon, \nabla\phi\rangle = \langle p_\varepsilon, \phi\rangle \qquad \forall \phi \in C_0^\infty(\mathbb{R}^m). \tag{4.8}$$

From $u_\varepsilon(x) = \int\limits_{\mathbb{R}^m} j_\varepsilon(x-y)u(y)dy$ we get

$$\partial_i\partial_k u_\varepsilon(x) = \int\limits_{\mathbb{R}^m} \partial_{x_i} j_\varepsilon(x-y)\partial_k u(y)dy.$$

Since $\operatorname{supp} j_\varepsilon \subset B_\varepsilon$ and if χ_ε denotes the characteristic function of B_ε, for $\varepsilon > 0$ the estimate

$$|\partial_{x_i} j_\varepsilon(x-y)| \le c(j)\varepsilon^{-m-1}\chi_{B_\varepsilon}(x-y)$$

is obvious. Then

$$\begin{aligned}|\partial_i\partial_k u_\varepsilon(x)| &\le c(j)\varepsilon^{-m-1}\int\limits_{\mathbb{R}^m} \chi_{B_\varepsilon}(x-y)|\partial_k u(y)|dy \\ &\le c(j)\varepsilon^{-m-1}|B_\varepsilon|^{\frac12}\left(\int\limits_{\mathbb{R}^m} \chi_{B_\varepsilon}(x-y)|\partial_k u(y)|^2 dy\right)^{\frac12}\end{aligned}$$

and therefore

$$\int\limits_{\mathbb{R}^m} |\partial_i \partial_k u_\varepsilon(x)|^2 dx \le c(j)^2 \varepsilon^{-m-2} c(m) \int\limits_{\mathbb{R}^m} |\partial_k u(y)|^2 \left(\int\limits_{\mathbb{R}^m} \chi_{B_\varepsilon}(x-y) dx \right) dy$$
$$\le c(j)^2 \varepsilon^{-2} c(m)^2 \|\nabla u\|^2,$$

where $c(m) = \omega_m \cdot m^{-1}$. This holds for $i, k = 1, \dots, m$. Therefore $\tilde{u}_\varepsilon \in D_B^2(\mathbb{R}^m)$. Because of (4.5) observing (4.8) and (4.2) we get for $\varepsilon, \varepsilon' > 0$

$$\begin{aligned} \|\tilde{u}_\varepsilon - \tilde{u}_{\varepsilon'}\|_2^2 &= \|\nabla \tilde{u}_\varepsilon - \nabla \tilde{u}_{\varepsilon'}\|^2 + \|\Delta \tilde{u}_\varepsilon - \Delta \tilde{u}_{\varepsilon'}\|^2 \\ &= \|\nabla u_\varepsilon - \nabla u_{\varepsilon'}\|^2 + \|p_\varepsilon - p_{\varepsilon'}\|^2 \to 0 \end{aligned}$$

$(\varepsilon, \varepsilon' \to 0)$ by the properties of the mollifier since $\nabla u \in L^2(\mathbb{R}^m)^m$ and $p \in L^2(\mathbb{R}^m)$. Therefore $(\tilde{u}_\varepsilon)$ is Cauchy in $D_B^2(\mathbb{R}^m)$ and there exists a unique $v \in D_B^2(\mathbb{R}^m)$ with $\|v - \tilde{u}_\varepsilon\|_2^2 \to 0$ as $\varepsilon \to 0$. This implies $\|\nabla v - \nabla u_\varepsilon\| = \|\nabla v - \nabla \tilde{u}_\varepsilon\| \to 0$. On the other hand, $\|\nabla u - \nabla u_\varepsilon\| \to 0$, $\|p - p_\varepsilon\| \to 0$ and from (4.7) we see that $v \in D_B^2(\mathbb{R}^m) \subset L_B^{1,2}(\mathbb{R}^m)$ is a further solution of (4.1). Then, by Lemma 4.1 $u = v \in D_B^2(\mathbb{R}^m)$. $\square$

Since $(D_B^2(\mathbb{R}^m), \|.\|_2)$ is complete (Theorem 2.1) an immediate consequence of Theorem 4.2 is

Corollary 4.3. $(L_\Delta^2(\mathbb{R}^m), \|.\|_{-1})$ *is a Banach space.*

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On Schauder Estimates for the Evolution Generalized Stokes Problem

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Abstract. This note is devoted to coercive estimates in anisotropic Hölder norms of the solution of the Cauchy–Dirichlet problem for the system of generalized Stokes equations arising in the linearization of equations of motion of a certain class of non-Newtonian liquids.

1. Formulation of main result

The paper is concerned with the initial-boundary value problem

$$\frac{\partial \boldsymbol{v}(x,t)}{\partial t} + \mathcal{A}(x,t,\frac{\partial}{\partial x})\boldsymbol{v}(x,t) + \nabla p(x,t) = \boldsymbol{f}(x,t), \tag{1.1}$$

$$\nabla \cdot \boldsymbol{v}(x,t) = 0, \quad x \in \Omega, \quad t \in (0,T),$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \qquad \boldsymbol{v}(x,t)|_{x\in S} = \boldsymbol{a}(x,t), \tag{1.2}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 2$, $S = \partial\Omega$, $\boldsymbol{v}(x,t) = (v_1,\dots,v_n)$ and $p(x,t)$ are unknown functions and $\mathcal{A}(x,t,\frac{\partial}{\partial x})$ is a second order strongly elliptic differential operator with the principal part $\mathcal{A}_0(x,t,\frac{\partial}{\partial x})$ satisfying the condition

$$C^{-1}|\xi|^2|\eta|^2 \leq \mathcal{A}_0(x,t,i\xi)\eta\cdot\eta \leq C|\xi|^2|\eta|^2 \tag{1.3}$$

for arbitrary $\xi,\eta \in \mathbb{R}^n$. If $\mathcal{A} = -\nu I\Delta$, then (1.1) is a well-known Stokes system.

It is assumed that the coefficients of the operator $\mathcal{A}$ are bounded and satisfy the Hölder condition with the exponent $\alpha \in (0,1)$ with respect to x and with the exponent $\alpha_1/2$, $\alpha_1 \in (\alpha,1)$, with respect to t, moreover, the leading coefficients have bounded derivatives with respect to spatial variables. The known functions $\boldsymbol{f},\boldsymbol{v}_0,\boldsymbol{a}$ should satisfy the necessary compatibility conditions, in the first line,

$$\nabla\cdot\boldsymbol{v}_0(x) = 0, \quad \boldsymbol{v}_0(x)|_{x\in S} = \boldsymbol{a}(x,0), \quad \int_S \boldsymbol{a}(x,t)\cdot\boldsymbol{n}(x)dS = 0, \tag{1.4}$$

where $\boldsymbol{n}(x)$ is the exterior normal to S and one more condition of a non-local character involving $\boldsymbol{a}_t$.

Let $p_0(x)$ be a solution of the Neumann problem

$$\nabla^2 p_0(x) = -\nabla\cdot\mathcal{A}(x,0,\frac{\partial}{\partial x})\boldsymbol{v}_0(x) + \nabla\cdot\boldsymbol{f}(x,0), \tag{1.5}$$
$$\frac{\partial p_0(x)}{\partial n}\Big|_{x\in S} = -\boldsymbol{n}\cdot\Big(\mathcal{A}(x,0,\frac{\partial}{\partial x})\boldsymbol{v}_0(x) - \boldsymbol{f}(x,0) + \boldsymbol{a}_t(x.0)\Big)$$

(if $\boldsymbol{v}$ does not possess the third order derivatives and $\boldsymbol{f}$ does not have the first order ones, the first equation in (1.5) should be understood in a weak sense). The compatibility condition reads

$$\boldsymbol{a}_t(x,0) + \mathcal{A}(x,0,\frac{\partial}{\partial x})\boldsymbol{v}_0(x) + \nabla p_0(x) = \boldsymbol{f}(x,0), \quad \forall x\in S. \tag{1.6}$$

Finally, we often assume that

$$\nabla\cdot\boldsymbol{f}(x,t) = 0, \qquad \boldsymbol{f}(x,t)\cdot\boldsymbol{n}(x)|_{x\in S} = 0 \tag{1.7}$$

or, what is the same thing, $\int_\Omega \boldsymbol{f}(x,t)\cdot\nabla\varphi(x)dx = 0$ for arbitrary smooth $\varphi(x)$. In this case the terms with $\boldsymbol{f}$ in (1.5) drop out.

Let l be a positive non-integral number: $l = [l]+\alpha$, $\alpha\in(0,1)$, and let $C^l(\Omega)$, $C^{l,l/2}(Q_T)$, $C^{l,l/2}(\Sigma_T)$ be standard Hölder spaces of functions (or vector fields) given in Ω, $Q_T = \Omega\times(0,T)$ and $\Sigma_T = S\times(0,T)$, respectively. We recall that the norms in $C^l(\Omega)$ and $C^{l,l/2}(Q_T)$ are given by

$$|u|_{C^l(\Omega)} = \sum_{0\le|j|\le[l]} \sup_\Omega |D^j u(x)| + [u]^{(l)}_\Omega,$$

where $j=(j_1,\dots,j_n)$, $|j| = j_1+\dots+j_n$, $D^j u(x) = \frac{\partial^{|j|}u(x)}{\partial x_1^{j_1}\dots\partial x_n^{j_n}}$ and

$$|u|_{C^{l,l/2}(Q_T)} = \sum_{0\le 2k+|j|\le[l]} \sup_{Q_T} |D^k_t D^j u(x,t)| + [u]^{(l,l/2)}_{Q_T}, \tag{1.8}$$
$$[u]^{(l)}_\Omega = \sum_{|j|=[l]} [D^j u]^{(\alpha)}_\Omega, \quad [v]^{(\alpha)}_\Omega = \sup_{x,y\in\Omega}\frac{|v(x)-v(y)|}{|x-y|^\alpha},$$
$$[u]^{(l,l/2)}_{Q_T} = [u]^{(l)}_{x,Q_T} + [u]^{(l/2)}_{t,Q_T},$$
$$[u]^{(l)}_{x,Q_T} = \sup_{t<T}[u(\cdot,t)]^{(l)}_\Omega, \quad [u]^{(l/2)}_{t,Q_T} = \sup_\Omega [u(x,\cdot)]^{(l/2)}_{(0,T)}.$$

These definitions extend in a standard way to the functions given on S and on Σ_T.

Proposition 1.1. *Let $S\in C^{2+\alpha}$, $\alpha\in(0,1)$, and let the operator $\mathcal{A}$ satisfy the above hypotheses. Assume also that $\boldsymbol{f}\in C^{\alpha,\alpha/2}(Q_T)$, $\boldsymbol{v}_0\in C^{2+\alpha}(\Omega)$, $\boldsymbol{a}\in C^{2+\alpha,1+\alpha/2}(\Sigma_T)$, where $Q_T=\Omega\times(0,T)$, $\Sigma_T = S\times(0,T)$, and that*

$$\sum_{k=1}^n |\mathcal{R}_k(\boldsymbol{a}_t\cdot\boldsymbol{n})|_{C^{\alpha,\alpha/2}(\Sigma_T)} < \infty,$$

where

$$\mathcal{R}_k(b) = -2\partial_k\int_S E(x-y)b(y)dS,$$

E being the fundamental solution of the Laplace equation:

$$E(x) = -\Gamma(n/2)|x|^{2-n}(2\pi^{n/2}(n-2))^{-1}, \;\; if \;\; n > 2,$$

$$E(x) = (2\pi)^{-1}\log|x|, \;\; if \;\; n = 2;$$

$\partial_k = \frac{\partial}{\partial x_k} - n_k\frac{\partial}{\partial n}$ being the kth component of the surface gradient ∇_S on S. Finally, let the conditions (1.4)–(1.7) *be satisfied. Then problem* (1.1), (1.2) *has a unique solution $(\boldsymbol{v}, p)$, $\boldsymbol{v} \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$, and it satisfies the inequality*

$$|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \leq c\Big(|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_T)} + |\boldsymbol{v}_0|_{C^{2+\alpha}(\Omega)} \tag{1.9}$$

$$+|\boldsymbol{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n} |\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})|_{C^{\alpha,\alpha/2}(\Sigma_T)}\Big).$$

The operators $\mathcal{R}_k$ that can be considered as the Riesz operators on S are continuous in the space $C^\alpha(S)$ but not in $C^{\alpha,\alpha/2}(\Sigma_T)$. However, the estimate (1.9) is coercive in the sense that its second term can be majorized by the first term multiplied by a certain constant. If (1.7) does not hold, then we can use the Weyl orthogonal decomposition

$$\boldsymbol{f} = \boldsymbol{f}_1 + \nabla\phi \equiv P_J\boldsymbol{f} + P_G\boldsymbol{f}$$

where ϕ is a solution of the Neumann problem

$$\Delta\phi(x) = \nabla \cdot \boldsymbol{f}(x), \quad x \in \Omega, \quad \frac{\partial\phi}{\partial n}\Big|_S = \boldsymbol{f}(x) \cdot \boldsymbol{n}(x), \tag{1.10}$$

so that $\boldsymbol{f}_1$ satisfies (1.7), and add $-\phi$ to $p(x,t)$. If $\boldsymbol{f}_1 = P_J\boldsymbol{f} \in C^{\alpha,\alpha/2}(Q_T)$ (which is not always the case), then the solution $\boldsymbol{v}, p_1 = p - \phi$ of the transformed problem exists and satisfies (1.9) with $P_J\boldsymbol{f}$ instead of $\boldsymbol{f}$. This implies the inequality

$$|\boldsymbol{v}|_{C^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C^{\alpha,\alpha/2}(Q_T)} \leq c\Big(|\boldsymbol{f}|_{C^{\alpha,\alpha/2}(Q_T)} + [P_G\boldsymbol{f}]^{(\alpha/2)}_{t,Q_T}$$

$$+|\boldsymbol{v}_0|_{C^{2+\alpha}(\Omega)} + |\boldsymbol{a}|_{C^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n} |\mathcal{R}_k(\boldsymbol{a}_t \cdot \boldsymbol{n})|_{C^{\alpha,\alpha/2}(\Sigma_T)}\Big)$$

that is a consequence of (1.9) and of the boundedness of the projectors P_G and P_J in $C^\alpha(\Omega)$:

$$|P_G f|_{C^\alpha(\Omega)} + |P_J f|_{C^\alpha(\Omega)} \leq c|f|_{C^\alpha(\Omega)}. \tag{1.11}$$

The proof of inequality (1.9) is based on the analysis of model problems, i.e., Cauchy and Cauchy–Dirichlet problem in the half-space for the system (1.1) with $\mathcal{A}(x, t, \frac{\partial}{\partial x}) = \mathcal{A}_0(\frac{\partial}{\partial x})$, whose solutions can be represented as linear combinations of some potentials. From these representation formulas estimates of solutions are derived, that are extended to the problem (1.1), (1.2) in a bounded domain via Schauder's localization procedure well known for elliptic and parabolic problems.

In the case of problem (1.1), (1.2) it is necessary to obtain an additional estimate for a mixed norm of the pressure function $p(x,t)$:

$$\langle p\rangle^{(\mu,\beta)}_{Q_T} = \sup_{t,t'\in(0,T)} |t-t'|^{-\beta}[p(\cdot,t)-p(\cdot,t')]^{(\mu)}_{x,\Omega}$$

for $\beta = \alpha_1/2$, $\alpha_1 \in (\alpha,1)$, $\mu = \mu_1 = 1+\alpha-\alpha_1 \in (0,1)$. If $\boldsymbol{a}_t\cdot\boldsymbol{n} = 0$, then $\nabla p = -P_G\mathcal{A}\boldsymbol{v}$, and this relation makes it possible to prove the inequality

$$\langle p\rangle^{(\mu_1,\alpha_1/2)}_{Q_T} = c\left(\sup_{t<T}|\boldsymbol{v}_t(\cdot,t)|_{C^{\alpha}(\Omega)} + \sup_{t<T}|\boldsymbol{v}(\cdot,t)|_{C^{2+\alpha}(\Omega)}\right) \tag{1.12}$$

and to estimate lower order norms $\sup_{Q_T}|p(x,t)|$ and $[p]^{(\alpha/2)}_{t,Q_T}$. For the Stokes problem (i.e., when $\mathcal{A} = -\nu I\nabla^2$), (1.9) was obtained by this method in [12] in the case $\boldsymbol{a}\cdot\boldsymbol{n} = 0$. The model problems are studied in [8, 9, 13, 17], L_p-estimates for problem (1.1), (1.2) are obtained in [15]. The detailed proof of Proposition 1.1 is given in [18].

Proposition 1.1 makes it possible to prove local solvability of the following nonlinear problem (see [8]):

$$\mathbf{v}_t - \nabla\cdot.D_\sigma(\sigma(\mathbf{v})) + (\mathbf{v}\cdot\nabla)\mathbf{v} + \nabla p = \mathbf{f}(x,t), \tag{1.13}$$

$$\boldsymbol{v}(x,0) = \boldsymbol{v}_0(x), \quad \boldsymbol{v}(x,t)|_{x\in S} = 0.$$

Here $D(\sigma)$ is a smooth convex function of symmetric real matrices $\sigma = (\sigma_{jk})_{j,k=1,\dots,n}$,

$$\sigma(\boldsymbol{v}) = \frac{1}{2}(\nabla\boldsymbol{v} + (\nabla\boldsymbol{v})^T) = \frac{1}{2}\Big(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial x_j}\Big)_{j,k=1,\dots,n}, \quad D_\sigma = (\frac{\partial D}{\partial\sigma_{jk}})_{j,k=1,\dots,n},$$

hence,

$$\nabla\cdot D_\sigma(\sigma(\boldsymbol{v})) = \Big(\sum_{k,q,r=1}^{n} D_{jk,qr}(\sigma(\boldsymbol{v}))\frac{\partial^2 v_q}{\partial x_k\partial x_r}\Big)_{j=1,\dots,n} \equiv -A(\sigma(\boldsymbol{v}),\frac{\partial}{\partial x})\boldsymbol{v},$$

$D_{jk,qr}(\sigma) = \frac{\partial^2 D(\sigma)}{\partial\sigma_{jk}\partial\sigma_{qr}}$, and the condition of convexity reads

$$C^{-1}\sum_{i,j=1}^{n}\kappa_{ij}^2 \le \sum_{j,k,q,r=1}^{n} D_{jk,qr}(\sigma)\kappa_{jk}\kappa_{qr} \le C\sum_{i,j=1}^{n}\kappa_{ij}^2$$

for arbitrary symmetric matrix $\kappa = (\kappa_{jk})_{j,k=1,\dots,n}$. Taking $\kappa_{jk} = \frac{1}{2}(\xi_j\eta_k + \xi_k\eta_j)$ we obtain

$$\frac{1}{2C}(|\xi|^2|\eta|^2 + (\xi\cdot\eta)^2) \le A(\sigma,i\xi)\eta\cdot\eta \le \frac{C}{2}(|\xi|^2|\eta|^2 + (\xi\cdot\eta)^2) \tag{1.14}$$

which is equivalent to ellipticity condition (1.3) for the operator $A(\sigma,\frac{\partial}{\partial x})$. The compatibility conditions for problem (1.13) have the form

$$\nabla\cdot\boldsymbol{v}_0(x) = 0, \quad \boldsymbol{v}_0(x)|_S = 0, \tag{1.15}$$

$$A(\sigma(\boldsymbol{v}_0),\frac{\partial}{\partial x})\boldsymbol{v}_0 + (\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0 + \nabla p_0(x) = \boldsymbol{f}(x,0),$$

where p_0 is a solution of the Neumann problem

$$\nabla^2 p_0(x) = -\nabla\cdot\Big((\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0 + A\big(\sigma(\boldsymbol{v}_0),\frac{\partial}{\partial x}\big)\boldsymbol{v}_0\Big),\quad x\in\Omega,$$

$$\frac{\partial p_0(x)}{\partial n}\Big|_S = -\boldsymbol{n}\cdot\Big((\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0 + A\big(\sigma(\boldsymbol{v}_0),\frac{\partial}{\partial x}\big)\boldsymbol{v}_0\Big),$$

in other words,

$$\nabla p_0 = -P_G\Big((\boldsymbol{v}_0\cdot\nabla)\boldsymbol{v}_0 + A\big(\sigma(\boldsymbol{v}_0),\frac{\partial}{\partial x}\big)\boldsymbol{v}_0\Big).$$

The following proposition is a consequence of Proposition 1.1 (see [18]):

Proposition 1.2. *Assume that $S\in C^{2+\alpha}$, $\alpha\in(0,1)$, $D(\sigma)$ is sufficiently smooth, and condition* (1.14) *is satisfied when σ belongs to the λ-neighborhood of the image of $\sigma(\boldsymbol{v}_0)$, i.e.,*

$$|\sigma-\sigma(\boldsymbol{v}_0(x))|\le\lambda$$

for some $x\in\Omega$. Then for arbitrary $\boldsymbol{f}\in C^{\alpha,\alpha/2}(Q_T)$, $\boldsymbol{v}_0\in C^{2+\alpha}(\Omega)$ satisfying (1.7), (1.15) *problem* (1.13) *has a unique solution* $(\boldsymbol{v},p)$, *$\boldsymbol{v}\in C^{2+\alpha,1+\alpha/2}(Q_\tau)$, $\nabla p\in C^{\alpha,\alpha/2}(Q_\tau)$, in a certain* (*small*) *time interval* $(0,\tau)$.

Problem (1.13), especially when $D_\sigma(\sigma(\boldsymbol{v}) = \nu(|\sigma(\boldsymbol{v})|)\sigma(\boldsymbol{v})$, and $\nu(r)$ is a power-like function, was studied by many authors. We address the reader to the book [10] and to the articles [1, 11] where global existence of a strong solution of this problem belonging to some Sobolev spaces is established.

The corresponding linear and nonlinear stationary problems are studied in [6].

2. On some generalizations and extensions of Proposition 1.1

Exterior domains. Proposition 1.1 can be extended to the case when Ω is an exterior domain with a compact boundary $S\in C^{2+\alpha}$, under the additional hypothesis concerning the operator $\mathcal{A}$:

$$\mathcal{A}\left(x,t,\frac{\partial}{\partial x}\right) = \nabla\cdot\ell\left(x,t,\frac{\partial}{\partial x}\right)$$

where ℓ is a first order operator:

$$\ell\left(x,t,\frac{\partial}{\partial x}\right) = \left(\sum_{q=1}^n \ell_{jk,mq}(x,t)\frac{\partial}{\partial x_q} + \ell_{jk,m}(x,t)\right)_{m=1,\dots,n},$$

so that

$$\nabla\cdot\ell\left(x,t,\frac{\partial}{\partial x}\right)\boldsymbol{u} = \left(\sum_{m=1}^n\frac{\partial}{\partial x_m}\Big(\sum_{k,q=1}^n \ell_{jk,mq}(x,t)\frac{\partial u_k}{\partial x_q} + \sum_{k=1}^n \ell_{jk,m}(x,t)u_k\Big)\right)_{j=1,\dots,n}$$

and

$$\mathcal{A}_0\left(x,t,\frac{\partial}{\partial x}\right)=\left(\sum_{k,q,m=1}^{n}\ell_{jk.mq}(x,t)\frac{\partial^2 u_k}{\partial x_q\partial x_m}\right)_{j=1,\dots,n}.$$

For the Stokes problem this extension was made in [13].

We do not assume that the data $\boldsymbol{f},\boldsymbol{v}_0$ or the solution of problem (1.1), (1.2) tend to zero as $|x|\to\infty$, which obliges us to define precisely operations P_G and P_J. We set $P_G=\nabla\phi$ where ϕ is a solution of exterior Neumann problem

$$\Delta\phi(x)=\nabla\cdot\boldsymbol{f}(x),\quad x\in\Omega,\quad \frac{\partial\phi}{\partial n}\Big|_S=\boldsymbol{f}(x)\cdot\boldsymbol{n}(x),\tag{2.1}$$

$$|\phi(x)|\le c|x|^{1+\alpha}\quad\text{for large }|x|,$$

$$\int_S\phi(x)dS=0.\tag{2.2}$$

Let us introduce the norm

$$||\boldsymbol{u}||_\alpha=\sup_{\Omega'}|\boldsymbol{u}(x)|+[\boldsymbol{u}]_\Omega^{(\alpha)},\tag{2.3}$$

where Ω' is a fixed bounded subdomain of Ω. It is clear that variation of Ω' leads to an equivalent norm. To be definite, we fix Ω' such that $\mathrm{dist}(\Omega\setminus\bar{\Omega}',S)\ge r_0>0$ and $|x-x_0|\le r_1$, $\forall x\in\Omega'$.

The following proposition holds.

Proposition 2.1. *If $\boldsymbol{u}$ has a finite norm* (2.3), *then problem* (2.1), (2.2) *has a solution satisfying the inequality*

$$||\nabla\phi||_\alpha\le c||\boldsymbol{u}||_\alpha.\tag{2.4}$$

If $\boldsymbol{u}=\nabla\cdot U\equiv\left(\sum_{k=1}^n\frac{\partial U_{ik}(x)}{\partial x_k}\right)_{i=1,\dots,n}$, then

$$||\phi||_\alpha\le c||U||_\alpha\equiv\max_{i,k=1,\dots,n}||U_{ik}||_\alpha.\tag{2.5}$$

Without presenting a detailed proof of this proposition, we note that the solution of problem (2.1), (2.2) can be defined as the sum

$$\phi(x)=\phi_1(x)+\phi_2(x)+\phi_0$$

where

$$\phi_1(x)$$
$$=\int_\Omega\left(\nabla_x E(x-y)-\nabla_{x_0}E(x_0-y)-\sum_{j=1}^n(x_j-x_{0j})\frac{\partial\nabla_{x_0}E(x_0-y)}{\partial x_{0j}}\right)\cdot\boldsymbol{u}(y)dy,$$

x_0 is a fixed point of a bounded domain $\Omega^c=\mathbb{R}^n\setminus\bar{\Omega}$, ϕ_2 is a solution of the problem

$$\Delta\phi_2(x)=0,\quad x\in\Omega,\quad \frac{\partial\phi_2}{\partial n}=\boldsymbol{u}\cdot\boldsymbol{n}-\frac{\partial\phi_1}{\partial n},\quad x\in S,$$

vanishing at infinity and $\phi_0=-\int_S(\phi_1(x)+\phi_2(x))dS$.

If $\boldsymbol{u} = \nabla \cdot U$, then we set $\phi_1(x) = \sum_{i=1}^n \frac{\partial R_i(x)}{\partial x_i}$,

$$R_i(x) = \sum_{k=1}^n \int_\Omega \Big(\frac{\partial E(x-y)}{\partial x_k} - \frac{\partial E(x_0-y)}{\partial x_{0k}} - \sum_{j=1}^n (x_j - x_{0j}) \frac{\partial^2 E(x_0-y)}{\partial x_{0j} \partial x_{0k}} \Big) U_{ik}(y) dy$$

$$- \sum_{k=1}^n \int_S E(x-y) U_{ik}(y) n_k(y) dS.$$

Inequalities (2.4), (2.5) follow from classical results of the theory of potentials.

By (2.4), operators P_G and P_J are bounded in the norm (2.3) but not in $C^\alpha(\Omega)$, since $\nabla\phi$ can grow at infinity.

The function $p(x,t)$ satisfies the relations

$$\Delta p = -\nabla \cdot \mathcal{A}\boldsymbol{v}, \quad x \in \Omega, \quad \frac{\partial p}{\partial n}\Big|_{x\in S} = -\boldsymbol{n} \cdot (\mathcal{A}\boldsymbol{v} + \boldsymbol{a}_t),$$

hence, $\nabla p = P_G \mathcal{A}\boldsymbol{v} - \nabla q$ where q is a solution of the problem

$$\Delta q = 0, \quad x \in \Omega, \quad \frac{\partial q}{\partial n}\Big|_{x\in S} = \boldsymbol{n} \cdot \boldsymbol{a}_t$$

vanishing at infinity. Inequality (2.5) makes it possible to estimate the Hölder constant of p by the norms of $\boldsymbol{v}$ and to prove inequality (1.12), but, in contrast to the case of bounded Ω, the estimate of $\sup_{Q_T} |p(x,t)|$ does not seem to be possible (but it is possible in the case of classical Stokes problem, because $\nabla \cdot \mathcal{A}\boldsymbol{v} = 0$ and p is a harmonic function).

Other arguments in the proof of (1.9), in particular those related to the Schauder localization procedure, apply to the case of exterior domains almost without changes.

Estimates in weighted Hölder norms. It is natural to try to minimize the number of compatibility conditions at the expense of condition (1.6) that is necessary for the existence of solution $\boldsymbol{v} \in C^{2+\alpha,1+\alpha/2}(Q_T)$, $\nabla p \in C^{\alpha,\alpha/2}(Q_T)$ of problem (1.1), (1.2) but has no physical meaning. This can be achieved by introducing the weight t^a into the Hölder norms.

Let $s \le l$, $Q'_t = \Omega \times (t/2, t)$. By $C_s^{l,l/2}(Q_T)$ we mean the space of functions with finite norm

$$|u|_{C_s^{l,l/2}(Q_T)} = \sup_{0<t<T} t^{(l-s)/2} [u]_{Q'_t}^{(l,l/2)} \tag{2.6}$$

$$+ \sum_{s<2k+|j|} \sup_{0<t<T} t^{(2k+|j|-s)/2} \sup_\Omega |D_t^k D_x^j u(x,t)| + |u|_{C^{s.s/2}(Q_T)}.$$

The derivatives $D_t^k D_x^j u(x,t)$ with $2k + |j| > s$ of the function $u \in C_s^{l,l/2}(Q_T)$ can have singularities at $t = 0$. The case $s \le 0$ is not excluded; if $s = 0$, then $|u|_{C^{s.s/2}(Q_T)} = \sup_{Q_T} |u(x,t)|$, and in the case $s < 0$ the last term in (2.6) is absent. The space $C_l^{l,l/2}(Q_T)$ coincides with $C^{l,l/2}(Q_T)$.

Expected (but not yet proved) extension of inequality (1.9) to the weighted spaces with a positive s has the form

$$|\boldsymbol{v}|_{C_s^{2+\alpha,1+\alpha/2}(Q_T)} + |\nabla p|_{C_{s-2}^{\alpha,\alpha/2}(Q_T)} \leq c\Big(|\boldsymbol{f}|_{C_{s-2}^{\alpha,\alpha/2}(Q_T)} + |\boldsymbol{v}_0|_{C^s(\Omega)} \tag{2.7}$$

$$+|\boldsymbol{a}|_{C_s^{2+\alpha,1+\alpha/2}(\Sigma_T)} + \sum_{k=1}^{n} |\mathcal{R}_k(\boldsymbol{a}\cdot\boldsymbol{n})|_{C_s^{2+\alpha,1+\alpha/2}(\Sigma_T)}\Big).$$

In the case $\boldsymbol{f}=0$, $\boldsymbol{a}=0$ this inequality implies

$$|\boldsymbol{v}(\cdot,t)|_{C^s(\Omega)} \leq c|\boldsymbol{v}_0|_{C^s(\Omega)};$$

in the case $s=0$ we would obtain the estimate of the maximum modulus of $\boldsymbol{v}(x,t)$ by the maximum modulus of $\boldsymbol{v}_0(x)$.

If $s<2$, then the compatibility condition (1.6) in general is meaningless, since the functions $\boldsymbol{a}_t(x,0)$, $\mathcal{A}(x,0,\frac{\partial}{\partial x})\boldsymbol{v}_0(x)$, $\nabla p_0(x,0)$, $\boldsymbol{f}(x,0)$ are not always well defined.

It seems that the arguments in the proof of (1.9) in [18] can be extended to the weighted spaces with $s>0$.

For parabolic initial-boundary value problems estimates in weighted Hölder norms are obtained in [2, 3, 4, 5, 7, 14] and in other papers.

The author hopes to return to problems discussed in this section in subsequent publications.

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Local Analyticity and Nonlinear Vector Fields

David S. Tartakoff

Abstract. We prove local analytic hypoellipticity for a quasilinear version of $\Box_b$ on the Heisenberg Group. The work is joint with A. Bove, M. Derridj and L. Zanghirati.

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Keywords. Quasilinear, analytic hypoellipticity, sums of squares, Heisenberg Group.

1. Introduction

In [13], Tartakoff and Zanghirati proved local analyticity of solutions u to equations $R_u u = f$ for nonlinear sums of squares of vector fields with prototype (in $\mathbb{R}^2$)

$$R_u = D_x^2 + x^{2r}(1 + h^2(x,t,u))D_t^2 = \sum_1^3 X_j^2$$

for $h(x,t,u)$ real analytic in its arguments and r arbitrary. The solution needed to start with minimal regularity, usually $X_j X_k u \in L^2$ for all j, k. Lower order terms in the X_j, with nonlinear but real analytic coefficients, could be added.

In [3], Derridj and Tartakoff proved a global analytic regularity result on a contact manifold M of real dimension $2n+1$ for some quasilinear equations. These were locally of the form $P_u u = f$ where in terms (for convenience) of the left invariant vector fields $X_1 \dots X_{2n}$ on the Heisenberg Group

$$P_u = \sum_{j,k} A_{jk}(x,u) X_j X_k + \sum_j A_j(x,u) X_j + A(x,u) \tag{1.1}$$

where the 'coefficients' A_{jk}, A_j, and A are (complex-valued) real analytic functions of the variables x and the function u and the matrix $A_{j,k}(x,u)$ is strictly positive definite for x near x_0, and u is a given function which again needed to be taken to be minimally smooth near x_0.

The reason that the first result above could be local and the second only global on a compact manifold proved to be only technical, although localizing high derivatives in the 'missing' direction T has always introduced significant additional complexity in the proofs.

2. Statement of results and *a priori* estimates

Our result is simple to state:

Theorem 2.1. *Let the operator P_u be given by* (1.1) *with real analytic coefficients and leading matrix $A_{j,k}(x,u)$ positive definite. Let u be a solution to the problem $P_u u = f$ in an open set Ω in $\mathbb{R}^{2n+1}$ with $f \in C^\omega(\Omega)$ and $u, X_j u, X_j X_k u \in L^2(\Omega)$. Then $u \in C^\omega(\Omega)$.*

Remark 2.2. The results of Xu ([15, 16]) in this situation show that the solution u is in fact $C^\infty(\Omega)$ so we may freely apply derivatives to u.

Remark 2.3. We believe that the same results holds for the case where the coefficients of the lower order terms in (1.1) depend on the $X_j u$.

As for *a priori* estimates, their derivation is as in the second work cited in the Introduction, and we merely state them, as they will not be unexpected.

Proposition 2.4. *The operator P_u satisfies the following maximal estimates: for any $\forall\, s \geq 0\, \exists\, C_s(u) : \forall v \in C_0^\infty(\Omega)$,*

$$\sum_{j=1}^{2n} \|X_j v\|_{H^s}^2 + \|v\|_{H^{s+1/2}}^2 \leq C_s\{(P_u v, v)_{H^s} + \|v\|_{H^s}^2\}$$

and

$$\sum_{j,k=1}^{2n} \|X_j X_k v\|_{H^s}^2 + \sum_{j=1}^{2n} \|X_j v\|_{H^{s+1/2}}^2 + \|v\|_{H^{s+1}}^2 \leq C_s\{\|P_u v\|_{H^s}^2 + \|v\|_{H^s}^2\}$$

The constants $C_s(u)$ depend on a bounded number of derivatives of the function u.

Remark 2.5. If we take $s = n+2$, H^s will be an algebra, which will be useful below.

3. Proof of the theorem. Localization of high powers of T

We write X_j' for the first n vector fields:

$$X_j' = X_j = \frac{\partial}{\partial x_j} - \frac{x_{n+j}}{2}\frac{\partial}{\partial t}, \quad j \leq n$$

and denote the others by X_j'' :

$$X_j'' = X_{n+j} = \frac{\partial}{\partial x_{n+j}} + \frac{x_j}{2}\frac{\partial}{\partial t}, \quad j \leq n$$

As always, we use the maximal estimate above with v replaced by an effective localization of $T^p u$, since once one has control over high T derivatives of the solution locally, the other derivatives will follow easily.

Observing that the naive localization $\varphi T^p u$ has very poor commutation properties: $[X, \varphi T^p]u = \varphi' T^p u$, a gain in the estimate of only 1/2 derivative while the localizing function φ suffers a whole derivative, leading to Gevrey class 2, we have proposed the definition

$$(T^p)_\varphi = \sum_{|\alpha+\beta|\le p} \frac{(-X')^\alpha X''^\beta \varphi}{\alpha!\beta!} X'^\beta X''^\alpha T^{p-|\alpha+\beta|}$$

in earlier work of this author ([10, 12]).

What we need about the vector fields at this point is $\forall j,k$:

$$[X'_j, X'_k] = [X''_j, X''_k] = [T, X'_j] = [T, X''_j] = 0,$$
$$[X'_j, X''_k] = \delta_{jk}T.$$

Then we have the commutation relations:

$$[X'_j, (T^p)_\varphi] \equiv 0 \mod \underline{C^p}\varphi^{(p+1)}X^p/p!$$

and

$$[X''_j, (T^p)_\varphi] \equiv (T^{p-1})_{T\varphi} \circ X''_j \mod \underline{C^p}\varphi^{(p+1)}X^p/p!$$

where underlining indicates how many such terms are present.

The commutation relations with functions are more complex and have been worked out in previous papers ([10, 12]). For this purpose it is convenient to make a simple change of variables so that the vector fields X' and X'' take a slightly simpler form ($x' = (x'_1, \ldots x'_n)$, $x'' = (x''_1, \ldots, x''_n) = (x_{n+1}, \ldots, x_{2n})$):

$$X'_j = \frac{\partial}{\partial x'_j} - x''_j \frac{\partial}{\partial t}, \qquad X''_j = \frac{\partial}{\partial x''_j}, \qquad j \le n. \tag{3.1}$$

The reason for this choice will be apparent below but it will reduce the number of subsidiary brackets that need to be considered.

In commuting $(T^q)_\varphi$ past coefficients g of the operator we may either compute $[(T^q)_\varphi, g]$ (which we do here) or treat a 'product' formula $(T^q)_\varphi(gw)$ and then consider only the terms where some derivatives fall on g.

Lemma 3.1. *With the above notation and a smooth function $g(x,t)$,*

$$[(T^q)_\varphi, g(x,t)] = \sum_{|\alpha+\beta|+\ell=0}^{q} \sum_{\substack{\alpha'+\alpha''=\alpha \\ \beta'+\beta''=\beta \\ 1\le \ell+|\alpha'+\beta'|}} \binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\binom{q-|\alpha+\beta|}{\ell} \tag{3.2}$$
$$\times \frac{(-1)^{|\alpha|}}{\alpha!\beta!}(X'^{\beta'}X''^{\alpha'}T^\ell g)(X'^{\alpha'}X'^{\alpha''}X''^{\beta''}X''^{\beta'}\varphi)X'^{\beta''}X''^{\alpha''}T^{q-|\alpha+\beta|-\ell}.$$

However, for the proper balance in $(T^*)_*$, we need to absorb some of the derivatives onto φ directly. The X'' derivatives are easy but the X' are not, although the above choice of the vector fields makes the situation a lot simpler.

In particular, letting $\psi = X''^{\beta'}\varphi$, since $X'X'' = X''\frac{\partial}{\partial x'} - x''X''\frac{\partial}{\partial t}$,

$$X'^{\alpha'+\alpha''}X''^{\beta''}\psi = \sum_{\alpha'''+\alpha^{iv}=\alpha'} \binom{\alpha'}{\alpha'''}(-x'')^{\alpha'''}X'^{\alpha''}X''^{\beta''}(\frac{\partial}{\partial t})^{\alpha'''}(\frac{\partial}{\partial x'})^{\alpha^{iv}}\psi$$

which we may write, with the conventions that now (x'') denotes either x'' or 1:

$$X'^{\alpha'+\alpha''}X''^{\beta'+\beta''}\psi = \underline{2^{|\alpha'|}}\,(x'')^{\alpha'}X'^{\alpha''}X''^{\beta''}D^{\alpha'}X''^{\beta'}\varphi. \tag{3.3}$$

To handle the binomial coefficients in (3.2) relating to the T derivatives, we cite a combinatorial result.

Proposition 3.2. *The following identity holds and may essentially be found in the book by Feller* [7]*:*

$$\binom{q-a-b}{\ell} = \sum_{t\le \ell,b}\binom{q-a-t}{\ell-t}(-1)^t\binom{b}{t}$$

Repeated use of the proposition, starting with $b=\alpha''_1$ and $a=|\beta+\alpha-\alpha''_1|$ and stripping off all the other α''_j and then β''_j we arrive, now with *multi*-indices τ_1,τ_2, at

Proposition 3.3.

$$\binom{q-|\alpha+\beta|}{\ell} = \sum_{\substack{\tau_1\le\alpha''\\ \tau_2\le\beta''\\ |\tau_1+\tau_2|\le\ell}}\binom{q-|\alpha'+\tau_1+\beta'+\tau_2|}{\ell-|\tau_1+\tau_2|}(-1)^{|\tau_1+\tau_2|}\binom{\alpha''}{\tau_1}\binom{\beta''}{\tau_2}$$

Thus the balance in (3.2) is restored if we we think of $\alpha',\beta',\tau_1,\tau_2$ as new indices subject to $\alpha'+\tau_1+\beta'+\tau_2\le q$ and use (3.3) once more to take care of the extra $\tau_1+\tau_2$ derivatives on φ. But instead of $\alpha',\beta',\tau_1,\tau_2$, we choose to use $\sigma_1=\alpha'+\tau_1,\sigma_2=\beta'+\tau_2$, with τ_1,τ_2, subject to $\sigma_1+\sigma_2\le q,\tau_1\le\sigma_1,\tau_2\le\sigma_2$.

We get

Proposition 3.4.

$$\binom{q-|\alpha+\beta|}{\ell}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\frac{1}{\alpha!\beta!} \tag{3.4}$$

$$= \sum_{\substack{\alpha'+\tau_1\le\alpha\\ \beta'+\tau_2\le\beta\\ \tau_1+\tau_2\le\ell}}\binom{q-|\alpha'+\tau_1+\beta'+\tau_2|}{\ell-|\tau_1+\tau_2|}\frac{(-1)^{|\tau_1+\tau_2|}}{\alpha'!\beta'!(\alpha''-\tau_1)!\tau_1!(\beta''-\tau_2)!\tau_2!}$$

$$= \sum_{\substack{\sigma_1=\alpha'+\tau_1\le\alpha\\ \sigma_2=\beta'+\tau_2\le\beta\\ \tau_1+\tau_2\le\ell\\ |\alpha+\beta|+\ell\le q}}\binom{q-|\sigma_1+\sigma_2|}{\ell-|\tau_1+\tau_2|}\frac{(-1)^{|\tau_1+\tau_2|}}{\alpha'!\beta'!(\alpha-\sigma_1)!\tau_1!(\beta-\sigma_2)!\tau_2!}.$$

Thus we may write

$$(T^q)_\varphi g(x,t)w(x,t) = \sum_{\substack{\ell+|\alpha+\beta|\le q\\ \alpha'+\alpha''=\alpha\\ \beta'+\beta''=\beta}}\binom{\alpha}{\alpha'}\binom{\beta}{\beta'}\binom{q-|\alpha+\beta|}{\ell}$$

$$\times\frac{(-1)^{|\alpha|}}{\alpha!\beta!}(X'^{\beta'}X''^{\alpha'}T^\ell g)(X'^{\alpha}X''^{\beta}\varphi)X'^{\beta''}X''^{\alpha''}T^{q-|\alpha+\beta|-\ell}w(x,t)$$

$$= \sum_{\substack{\sigma_1=\alpha'+\tau_1\le\alpha \\ \sigma_2=\beta'+\tau_2\le\beta \\ \tau_1+\tau_2\le\ell \\ |\alpha+\beta|+\ell\le q}} \frac{(-1)^{|\tau_1+\tau_2|+|\alpha'|+|\tau_1|}(-1)^{|\alpha''-\tau_1|}}{\alpha'!\beta'!(\alpha''-\tau_1)!\tau_1!(\beta''-\tau_2)!\tau_2!}$$

$$\times(X'^{\beta'}X''^{\alpha'}T^\ell g)(X'^{\sigma_1}X'^{\alpha''-\tau_1}X''^{\beta''-\tau_2}X''^{\sigma_2}\varphi)$$

$$\times X'^{\tau_2}X'^{\beta''-\tau_2}X''^{\alpha''-\tau_1}X''^{\tau_1}T^{q-r-\ell}w(x,t)$$

$$= \sum_{\substack{\sigma_1=\alpha'+\tau_1\le\alpha \\ \sigma_2=\beta'+\tau_2\le\beta \\ \tau_1+\tau_2\le\ell \\ |\alpha+\beta|+\ell\le q}} \binom{q-|\sigma_1+\sigma_2|}{\ell-|\tau_1+\tau_2|} \frac{(-1)^{|\tau_1+\tau_2+\sigma_1|}(-1)^{|\alpha-\sigma_1|}}{(\sigma_1-\tau_1)!(\sigma_2-\tau_2)!(\alpha-\sigma_1)!\tau_1!(\beta-\sigma_2)!\tau_2!}$$

$$\times(X'^{\sigma_2-\tau_2}X''^{\sigma_1-\tau_1}T^\ell g)(X'^{\sigma_1}X'^{\alpha-\sigma_1}X''^{\beta-\sigma_2}X''^{\sigma_2}\varphi)$$

$$\times X'^{\tau_2}X'^{\beta-\sigma_2}X''^{\alpha-\sigma_1}X''^{\tau_1}T^{(q-\ell-|\sigma_1+\sigma_2|)-|\alpha-\sigma_1+\beta-\sigma_2|}w(x,t)$$

so that, using (3.3),

$$(T^q)_\varphi g(x,t)w(x,t) = \sum_{\substack{\tau_1\le\sigma_1,\tau_2\le\sigma_2 \\ \ell+|\sigma_1+\sigma_2|\le q}} \frac{\pm X'^{\sigma_2-\tau_2}X''^{\sigma_1-\tau_1}T^\ell g}{(\sigma_2-\tau_2)!(\sigma_1-\tau_1)!} \tag{3.5}$$

$$\times\frac{(x''+1)^{\tau_1}}{\tau_1!\tau_2!}(T^{q-\ell-|\sigma_1+\sigma_2|})_{X'^{\tau_2};\varphi^{(|\sigma_1+\sigma_2|)}}\circ X''^{\tau_1}w(x,t),$$

where $\pm$ denotes $(-1)^{|\tau_1+\tau_2+\sigma_1|}$. Here the new notation $(T^{\tilde q})_{X'^\tau;\tilde\varphi}$ denotes $X'^\tau\circ(T^{\tilde q})_{\tilde\varphi}$ except that the additional vector fields X'^τ occur to the right of $\tilde\varphi$ but to the left of the other vector fields in $(T^{\tilde q})_{\tilde\varphi}$. The trick is in moving them to the left of $\tilde\varphi$ without incurring unacceptable numbers of derivatives on $\tilde\varphi$ without corresponding compensation.

We point out, though, at this point the essential features of this expression. The function $g(x,t)$ that has been moved to the left of the localization of T^p has derivatives which are compensated either by the corresponding factorials in the denominator or by a reduction in order of powers of T. The extra copies of X which enter are balanced by the remaining factorials, and the derivatives on the localizing function φ.

Now in (3.5), when we do commute the extra X' derivatives to the left of the expression for $(T^{q'})_{\varphi'}$, all that happens is that for each such extra X' we obtain two terms, one in which X' is composed on the left with the whole expression $(T^{q'})_{\varphi'}$ and the other where this X' lands on φ', i.e., to the left of all other derivatives on φ', and yet by means of the special form of the vector fields X' and X'', these are not harmful, since X' is a sum of two terms, a derivative $\partial/\partial x'$ which passes directly onto φ' past products of copies of X' and X'', and the expression $x''\partial/\partial t$ which may be split – the x'' staying to the very left and $\partial/\partial t$ which goes straight onto φ' as has already occurred in obtaining (3.5). The result is more terms of the type that already appear, with coefficients $(x''+1)$ raised to the power $\tau_2'\le\tau_2$

and the corresponding derivatives on φ'. That is, writing $\tau_2 = \tau_2' + \tau_2''$, we have the more amenable form of (3.5):

$$(T^q)_\varphi g(x,t)w(x,t) = \sum_{\substack{\tau_1\le\sigma_1,\tau_2\le\sigma_2\\ \ell+|\sigma_1+\sigma_2|\le q\\ \tau_2=\tau_2'+\tau_2''}} \frac{\pm X'^{\sigma_2-\tau_2}X''^{\sigma_1-\tau_1}T^\ell g}{(\sigma_2-\tau_2)!(\sigma_1-\tau_1)!}\,\frac{(x''+1)^{\tau_1+\tau_2'}}{\tau_1!\tau_2!} \tag{3.6}$$

$$\times X'^{\tau_2''}\circ (T^{q-\ell-|\sigma_1-\tau_1+\sigma_2-\tau_2|-|\tau_1+\tau_2|})_{\varphi^{(|\tau_1+\tau_2+\tau_2'|)}}\circ X''^{\tau_1}w(x,t).$$

Remark 3.5. Virtually none of this analysis is needed in the case when the coefficients g are independent of t (the rigid case), for the alternating formula and the skewed binomial are absent then.

4. Conclusion. Nonlinear coefficients

The proof of analyticity focuses on bounding high T derivatives of the solution u, localized as in $(T^p)_\varphi u$, in L^2 norm, by $C^p p!$ To profit optimally from the a priori estimate there should also be one good vector field X to the left. In evaluating $(P_u(T^p)_\varphi u, (T^p)_\varphi u)_{L^2}$, which appears on the right of the a priori estimate, we must commute

$$[(T^p)_\varphi, P_u],$$

(with $P_u = \sum_{j,k} A_{jk}(x,t,u)X_jX_k + \sum_j A_j(x,t,u)X_j + A(x,t,u)$). We have seen above the effect of the bracket with the X_j and also with coefficients, though it is less clear how to proceed with the complex expressions on the right-hand side of the last proposition. Actually the bounds when the coefficients $A_{(j)(k)}$ are independent of u are just those of the author's papers [10, 11] and, while not simple, will not be reproduced here.

To handle the case when the functions $g = A(u,x,t)$ and w depend on the solution u as well, the above formula remains valid, but we need to understand the derivatives of these functions somewhat better. In particular, since the solution u will appear both in the coefficient $g = A$ and also in the form of the function $w = u$ and we need to be sure that when a large number of T derivatives land on one of these copies of u, it is only in the precisely balanced form T^q_Ψ.

The first observation is that the above formula could just as well have had the roles of g and w reversed.

Thus if we denote by H_w (resp. H_g) (H for 'high' and R for 'remainder') the portion of the sum in (3.5) in which w (resp. g) is subjected to *more than* $q/2$ derivatives *in the grouping* $X'^{a'}(T^{\tilde q})_\Psi X''^{a''}$ for some $\tilde q, a', a''$ and Ψ, we may write

$$(T^q)_\varphi gw = H_w + R_w = H_g + R_g. \tag{4.1}$$

Note that there are no terms which appear in *both* H_w *and* H_g, since each of the $(T^q)_\Psi$ is homogeneous and the total order of differentiation on the right-hand side certainly does not exceed q. Thus

$$H_w \subset R_g \text{ and } H_g \subset R_w,$$

which means that

$$(T^q)_\varphi gw = H_w + H_g + R_{g,w} \tag{4.2}$$

where the terms in $R_{g,w}$ have at most $q/2$ derivatives on g *and* at most $q/2$ derivatives on w.

In other words, and with slightly more detail,

$$\begin{aligned}(T^q)_\varphi gw = &\sum C^R_{a,b}(D^a g)(D^b w) \\ &+ \sum C^{H_w}_{a,b,q'}(D^a g)({X'}^{b_1}(T^{q'})_{\varphi^{(b_2)}}{X''}^{b_3} w) \\ &+ \sum C^{H_g}_{a,b,q'}(D^a w)({X'}^{b_1}(T^{q'})_{\varphi^{(b_2)}}{X''}^{b_3} g).\end{aligned} \tag{4.3}$$

For bounds on the constants here we shall refer back to (3.6); here we note that in both sums, $|a| \leq q/2$.

Now the derivatives, denoted $D^a g, D^b w, D^a w$ above, which do not have a 'balance' (i.e., where there is no $T^{\tilde{p}}_{\tilde{\varphi}}$), have been thoroughly treated in the paper by Tartakoff and Zanghirati [13] where operators of the form

$$D_x^2 + x^{2r}(1 + h^2(x,t,u))D_t^2$$

were treated, permitting less esoteric localization. There the derivatives of $g = A(u,x,t)$, were written as products of derivatives of copies of u in a form which could be understood by the Faà di Bruno formula, somewhat rederived.

When the derivatives appearing on the nonlinear function are of the special form $(T^{q'})_{\varphi'}$, we must be careful to preserve that form for the highest order (T) derivatives at least.

But this is precisely what has just been accomplished by the discussion above. For instance, where $g = A(u,x,t)$ receives a lot of derivatives, we may treat g as a product in the sense that if $b_3 \neq 0$ in (4.3) then we apply one derivative to g of the form X'' and, of course, obtain a product of the form $A'(u,x,t)u'$ and from then on proceed with $u' = w$ and $A' = g$. If $b_3 = 0$, then we may always strip off one T from $(T^{q'})_{\tilde{\varphi}}$:

$$(T^{q'})_{\tilde{\varphi}} \equiv (T^{q'-1})_{\tilde{\varphi}} \circ T \mod C^{q'} \tilde{\varphi}^{(q'+1)} X^{q'}/q'!$$

and proceed as above applying the separate T to $A(u,x,t)$.

Thus after taking the L^2 norm of (4.3), perhaps with another X since $(P_u(T^p)_\varphi u, (T^p)_\varphi u)$ involves $C\|X[A(x,t,u),(T^p)_\varphi]u\|^2$, in each term on the right we have the L^2 norm of a product (after Faà di Bruno) of low order terms H_j not involving the localizing function and one, possibly higher order term, of the precise form ${X'}^{b_1} T^{\tilde{p}}_{\tilde{\varphi}} {X''}^{b_2}$.

Hence, taking the product of L^∞ norms of the low order terms and the L^2 norm of the last,

$$\|X \prod_j H_j(x,t,u) {X'}^{b_1} T^{\tilde{p}}_{\tilde{\varphi}} {X''}^{b_2}\|_{L^2}$$

$$\leq C \prod_j \|(X) H_j(x,t,u)\|_{L^\infty(supp\ \varphi)} \|(X){X'}^{b_1} T^{\tilde{p}}_{\tilde{\varphi}} {X''}^{b_2} u\|_{L^2}$$

By writing (X) here we mean that an X (*the* X) will appear in one of the locations shown, or will perhaps have given one more derivative inside one of the H_j. For the low order terms, in L^∞ norms, we have seen in [13] that one more derivative may be allowed per term; and in the principal term, the maximal estimate permits the one X shown.

The 'leading' term (here written last) is resubjected to the *a priori* estimate (repeatedly) until its order drops to $p/2$.

And for the terms in the product of low order, we pass to L^2 norms of slightly higher numbers of derivatives (by the Sobolev embedding) and proceed as before, with new localizing functions geared to the number of derivatives present. But all of these choices and calculations are precisely what has been worked out in the paper of the last two authors and there are no essential differences in the present case, since we are able to preserve the form of $T_{\tilde{\varphi}}^{\tilde{p}}$ when the order is still high.

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Hyperbolic Problems and Regularity Questions
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Strongly Hyperbolic Complex Systems Reduced Dimension, Hermitian Systems

Jean Vaillant

1. Introduction

We consider a first order system:

$$a(D) = ID_0 + \sum_1^n a_k D_k \, ,$$

where I is the identity matrix and a_k is a complex-valued $m \times m$ matrix.

Let $a(\xi)$ be the principal symbol of $a(D)$:

$$a(\xi) = I\xi_0 + \sum_1^n a_k \xi_k \, .$$

We define the *reduced dimension* of a:

$$d(a) = \begin{array}{l}\text{dimension of the real vector space of } M(C) \text{ generated by} \\ (I, \dots, \operatorname{Re} a_k, \dots, \operatorname{Im} a_k, \dots),\ 1 \leq k \leq n.\end{array}$$

We have stated in [10, 11, 12] that:

$$d(a) = d(T^{-1}aT) \quad \text{where } T \text{ is invertible} \in M(C)\, ; \quad d({}^t a) = d(\bar{a}) = d(a)\, .$$

Also, if a is diagonalizable, $d(a) = d(\det a)$.

We recall the theorem by Kasahara, Yamaguti: a is strongly hyperbolic if and only if a is uniformly diagonalizable; that means:

$$\forall \xi \text{ the zeroes in } \tau \text{ of } \det\bigl(\tau I + a(\xi)\bigr) = 0 \text{ are real;}$$

in other words if we denote: $p(\xi') = \sum_1^n a_k \xi_k$,

i) $\forall \xi'$ the eigenvalues of $p(\xi')$ are real.

ii) $p(\xi')$ is diagonalizable; if τ is an eigenvalue of multiplicity μ the dimension of the corresponding vector space of eigenvectors is μ;

so there exists $\Delta(\xi')$ such that:

$$\Delta^{-1}(\xi') p(\xi') \Delta(\xi') \text{ is diagonal } \forall \xi'.$$

iii) there exists $\Delta(\xi')$, such that (uniformly):

$$\|\Delta(\xi')\| \leq M\,, \quad |\det \Delta(\xi')| \geq \varepsilon > 0\,.$$

Lemma 1.1. *We denote by ϕ^i_j the entries of the matrix a. We assume a is diagonalizable and we denote:*

V = vector space spanned by the real linear forms: $\operatorname{Re}\phi^i_j$, $\operatorname{Im}\phi^i_j$, $i > j$.

Then:

i) $1 \leq i \leq m: \quad \phi^i_i(\xi) = \xi_0 + \chi_i(\xi') + i\lambda_i(\xi'), \quad \chi_i,\ \lambda_i$ *real-valued,* $\lambda_i \in V$.
ii) $p < q, \quad \operatorname{Re}\phi^p_q \in V, \quad \operatorname{Im}\phi^p_q \in V$.

Proof. See [1, 6]. □

Consequence. $d(a) \leq 2m(m-1)/2 + m = m^2$.

The proof of the theorem is divided into four parts, according to the number of forms of a basis of V. In this paper we study two cases. In Section 2, $d(V) = m^2 - m - 3$; in Section 3 it is $m^2 - m - 2$. The other cases will be published in another article [13].

We state the

Theorem 1.2. *If $m \geq 4$, if $d(a) \geq m^2 - 3$ and if a is strongly hyperbolic, then a is prehermitian, that means:*

$$\exists T \in M(C), \text{ such that: } T^{-1}a(\xi)T \text{ is hermitian } \forall\xi\,.$$

Remark 1.3.

1. If $m = 4$, the theorem was stated in [12].
2. If we replace the assumption of uniform diagonalizability by the weaker one of diagonalizability and the assumption of reduced dimension by the stronger: $d(a) \geq m^2 - 2$, $(m \geq 3)$, see [10, 11], the theorem is yet valuable; so here we study only the case $m^2 - 3$.
   The result in the case $m = 2$ was obtained in [5] without assumption of dimension.
3. The real case and the extension of the results to the case of variable coefficients were considered in [6, 1, 7, 8, 9, 2, 3, 4].

Lemma 1.4. *Denote by E_j the $m \times m$ matrix in which all the non diagonal elements are 0, all the diagonal elements are equal to 1 except the (j, j) element equal to $-i$, $(i^2 = -1)$. Then:*

$$E_j^{-1}(\phi)E_j = \begin{pmatrix} \phi^1_1 & & & \\ & & -i\phi^j_k & \\ & i\phi^k_j & \phi^k_k & \\ & & & \phi^m_m \end{pmatrix}.$$

Lemma 1.5. *If b is prehermitian, there exists a hermitian positive definite matrix H such that:*

$$bH = H\,{}^t\overline{b}\,.$$

Proof. There exists T such that:

$$T^{-1}bT = \overline{{}^t(T^{-1}bT)}\,,$$

and we denote:

$$H = T^t\overline{T}\,. \qquad \square$$

We assume now: $m \geq 5$.

2. Dimension of the space $d(V) = m^2 - m - 3$

Notations. The m diagonal forms are linearly independent; we choose $m-1$ of the $\chi_j(\xi')$ as coordinates χ_j and the mth one equal to 0. $m(m-1)-3$ among the forms: $\operatorname{Re}\phi^i_j$, $\operatorname{Im}\phi^i_j$, $i>j$ are chosen as coordinates: ξ^i_j, η^i_j; we denote: $z^i_j = \xi^i_j + i\eta^i_j$ and by z the set of these forms; a dependent form of V is a linear function of the elements of z: $\phi(z)$.

$$a(\xi) = \begin{pmatrix} \delta_1 & & & & & & & \phi^p_q(z)_{p<q} \\ & \ddots & & & & & & \\ & & \delta_{k-1} & & & & & \\ \phi^i_j(z)_{i>j} & & & \xi_0 + i\lambda_k(z) & & & & \\ & & & & \delta_{k+1} & & & \\ & & & & & \ddots & & \\ & & & & & & & \delta_m \end{pmatrix}$$

with $\delta_j = \xi_0 + \chi_j + i\lambda_j(z)$.

Lemma 2.1. $\lambda_k(z) = 0\ \forall k$.

Proof. $\det p(\xi')$ is real; the coefficient of $\Pi_{j\neq k}\chi_j$ is $\lambda_k(z)$. $\square$

Lemma 2.2. *Let $p<q$. If $\operatorname{Re}\phi^q_p$ and $\operatorname{Im}\phi^q_p$ are linearly independent, then:*

$$\phi^p_q = k^p_q\overline{\phi}^q_p\,, \quad k^p_q \in \mathbb{R}\,.$$

Proof. We choose $\chi_q = 0$; we consider the coefficient of $\Pi_{i\neq p,q}\chi_i$ in $\det p(\xi')$; we get:

$$\phi^p_q(\xi')(\xi^q_p + i\eta^q_p) \quad \text{are real}$$

and we obtain the result. $\square$

Lemma 2.3. *Let $p<q<r$. Assume that 2 couples among*

$$\left(\operatorname{Re}\phi^q_p, \operatorname{Im}\phi^q_p\right)\,, \quad \left(\operatorname{Re}\phi^r_p, \operatorname{Im}\phi^r_p\right)\,, \quad \left(\operatorname{Re}\phi^r_q, \operatorname{Im}\phi^r_q\right)$$

are formed by independent linear forms and assume the elements of the third couple do not depend on the elements of one of the couples of independent forms, then:

i) *if the elements of the third couple are also linearly independent forms:*

$$k_r^p = k_q^p k_r^q; \tag{2.1}$$

ii) *if* $\operatorname{Re}\phi_p^q$, $\operatorname{Im}\phi_p^q$, $\operatorname{Re}\phi_q^r$, $\operatorname{Im}\phi_q^r$ *are linearly independent:*

$$\phi_r^p = k_q^p k_r^q \overline{\phi_p^r}; \tag{2.2}$$

if $\operatorname{Re}\phi_p^q$, $\operatorname{Im}\phi_p^q$, $\operatorname{Re}\phi_p^r$, $\operatorname{Im}\phi_p^r$ *are linearly independent:*

$$k_r^p \overline{\phi_q^r} = k_q^p \phi_r^q; \tag{2.3}$$

if $\operatorname{Re}\phi_q^r$, $\operatorname{Im}\phi_q^r$, $\operatorname{Re}\phi_p^r$, $\operatorname{Im}\phi_p^r$ *are linearly independent*

$$k_r^q \overline{\phi_q^p} = k_r^p \phi_p^q. \tag{2.4}$$

Proof. Assume $\operatorname{Re}\phi_p^q$, $\operatorname{Im}\phi_p^q$, $\operatorname{Re}\phi_q^r$, $\operatorname{Im}\phi_q^r$ are linearly independent. We choose $\chi_r = 0$ and we consider the coefficient of $\Pi_{i\neq p,q}\chi_i$ in $\det p(\xi')$. We obtain:

$$\left(\overset{q}{\xi_p} + i\eta_p^q\right)\left(\overset{r}{\xi_q} + i\eta_q^r\right)\left(\overset{p}{\phi_r} - k_q^p k_r^q \overline{\phi_p^r}\right) \quad \text{is real}\,. \tag{2.5}$$

The i) is evident. If $\operatorname{Re}\phi_r^p$, $\operatorname{Im}\phi_r^p$ do not depend on $\overset{q}{\xi_p}, \eta_p^q$, we obtain:

$$\left(\xi_q^r + i\eta_q^r\right)\left(\overset{p}{\phi_r} - k_q^p k_r^q \overline{\phi_p^r}\right) = 0\,,$$

and the result. □

Proof of Theorem 1.2. We proceed by induction and direct use of the Lemmas. Thanks to Lemmas 2.2, 2.3, the total number of ϕ_j^i such that $\operatorname{Re}\phi_j^i$ or $\operatorname{Im}\phi_j^i$ are dependent is less or equal than $2\times 3 = 6$; the other ϕ_j^i have the form: $\xi_j^i + i\eta_j^i = z_j^i$ and $k_i^j\left(\xi_j^i - i\eta_j^i\right)$, $i > j$.

The relations (2.1) are verified.

The maximum number of rows in which all the real and imaginary parts of the elements are linearly independent is $m-3$.

If we are in this case, we can reduce to the case in which these rows are the last $m-3$ ones. We denote by $*$ the possible ϕ_j^i which will depend on the independent terms, $i > j$.

$$a(\xi) = \begin{pmatrix} \xi_0+\chi_1 & . & . & . & . & . & . & k_m^1\overline{z_1^m} \\ * & \xi_0+\chi_2 & . & . & . & . & . & . \\ * & * & \xi_0+\chi_3 & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ z_i^m & z_i^m & z_3^m & . & . & . & . & \xi_0+\chi_m \end{pmatrix}.$$

Let $z_1^m = z_2^m = \cdots = z_{m-1}^m = 0$; the submatrix $(m-1)\times(m-1)$ denoted by b and formed by the first $m-1$ rows and columns of a, the elements of which are the restricted ϕ_j^i, $1 \le j < i \le m-1$ is uniformly diagonalizable; its reduced dimension is $(m-1)^2-3$; by induction, it is prehermitian; by Lemma 1.5 there exists H such that:

$$bH = H^t\overline{b}.$$

Identifying in this formula the terms in χ_k, we obtain that H is diagonal and that:

$$1 \le p < q \le m-1\,, \quad \phi_q^p = k_q^p \overline{\phi}_p^q\,, k_q^p > 0\,,$$
$$k_q^p = k_r^p k_q^r\,, \quad p < r < q.$$

By Lemma 2.3, we obtain:

$$k_m^{m-2} = k_{m-1}^{m-2} k_m^{m-1}\,, \ldots, k_m^1 = k_{m-1}^1 k_m^{m-1}\,.$$

Let:

$$z_1^{m-1} = z_2^{m-1} = \cdots = z_{m-2}^{m-1} = z_1^m = z_2^m = z_{m-2}^m = 0\,.$$

We obtain $k_m^{m-1} > 0$ and finally:

$$k_q^p > 0\,, \qquad p < q \quad \text{and} \quad k_q^p = k_r^p k_q^r\,, \quad \text{if } p < r < q\,.$$

We transform a by the matrix

$$K = \begin{pmatrix} 1 & & & & & & & \\ & \frac{1}{\sqrt{k_2^1}} & & & & & & \\ & & \cdot & & & & & \\ & & & \cdot & & 0 & & \\ & & & & \frac{1}{\sqrt{k_q^1}} & & & \\ & & & 0 & & \cdot & & \\ & & & & & & \cdot & \\ & & & & & & & \frac{1}{\sqrt{k_m^l}} \end{pmatrix}$$

$K^{-1}aK$ is hermitian.

If the number of rows of independent terms is $m-4$, the number of rows, some elements of which contain dependent terms is 4, we can get the rows of independent terms as the last rows. As before, we let:

$$z_1^m = \cdots = z_{m-1}^m = 0$$

and we obtain the same results. We use Lemma 2.3. We can choose the next to last row such that there is at most one dependent form in this row; we let the other forms of this row to be equal to zero and cancel the dependent form if it is not identically 0, we express one variable as a linear function of the other ones; we get the same result as before.

If the number of rows of independent terms is less or equal than $m-5$, the number of rows of dependent terms is greater or equal than 5.

We distinguish the cases $m=5$ and $m>5$. We remark also that we can use the used argument for columns as we used for rows.

For $m=5$, the number of rows with dependent terms is 5. In the last row, there is one dependent term, it depends only on the coordinates in the row; otherwise we can come back to the preceding cases by change of variables. Then we cancel the last row and continue as before, with some adaptations.

For $m > 5$, either we come back to the preceding cases or we adapt easily the previous arguments.

3. Dimension of the space $d(V) = m^2 - m - 2$

Notations. $m-1$ diagonal forms are linearly independent; we can choose $m-2$ of the $\chi_i(\xi')$ as coordinates and one equal to 0.

(We can change the choice of the missing χ_k):

$m(m-1)-2$ among the forms $\operatorname{Re}\phi^i_j$ and $\operatorname{Im}\phi^i_j$ are chosen as coordinates (ξ^i_j, η^i_j); $z^i_j = \xi^i_j + i\eta^i_j$; $z = \{\xi^i_j\,,\ \eta^i_j\}$

$$a(\xi) = \begin{pmatrix} \xi_0 + \sum\limits_{\substack{2\le j\le m\\ j\neq k}} c_j\chi_j + \psi(z) + i\lambda_1(z) & & & & & & \\ & \delta_2 & & & \phi^p_q(z)_{p<q} & & \\ & & \ddots & & & & \\ & & & \delta_{k-1} & & & \\ & \phi^i_j(z)_{i>j} & & & \xi_0 + i\lambda_k(z) & & \\ & & & & & \delta_{k+1} & \\ & & & & & & \ddots \\ & & & & & & & \delta_m \end{pmatrix}.$$

Lemma 3.1.

i) *Assume $c_j \neq 0$, then $\lambda_k = 0\ \forall k \neq 1, j$.*

ii) *Assume $\chi_k = 0$ and $c_j = 0\ \forall j$, then $\lambda_l = 0\ \forall l \neq 1, k$.*

Proof. i) The proof is almost analogous to the one of Lemma 2.1. We consider in $\det p(\xi')$, the coefficient of

$$\Pi\chi_2 \dots \chi_j^2 \dots \widehat{\chi}_k \dots \chi_m\,; \quad 2 \le k \le m\,,$$

where $\widehat{}$ means the term is missing, and we obtain $\lambda_k = 0$.

ii) We let:

$$\xi_0 = \xi_0' + \chi_k\,, \quad \chi_l + \chi_k = \chi_{l'} \quad l \neq k\,,$$

and we proceed as before, replacing χ_j^2 by χ_k^2. □

Lemma 3.2. *Assume $c_j \neq 0$, $1 < p < q$. Assume that $\operatorname{Re}\phi^q_p$, $\operatorname{Im}\phi^q_p$ are linearly independent forms of V and $p \neq j$, $q \neq j$, then $\phi^p_q = k^p_q\overline{\phi^q_p}$.*

The proof is analogous to the one of Lemma 2.2 (cf. also the proof of Lemma 3.1).

Lemma 3.3.

i) *Assume $c_j \neq 0$, $1 < p < q < r$; $p \neq j$, $q \neq j$, $r \neq j$.*
Assume that 2 couples among

$$(\operatorname{Re}\phi^r_p, \operatorname{Im}\phi^r_p), (\operatorname{Re}\phi^r_q, \operatorname{Im}\phi^r_q), (\operatorname{Re}\phi^q_p, \operatorname{Im}\phi^q_p)$$

are formed by independent linear forms, and that the elements of the third couple do not depend on the elements of one of the couples of independent *forms, then we have the same results as in the Lemma* 2.3*; we denote them* (3.1), (3.2), (3.3), (3.4).

ii) *Assume* $\chi_i = 0$, $c_j = 0\ \forall j$, $1 < p < q < r$, $p \neq k$, $q \neq k$, $r \neq k$, *we obtain the same results.*

Proof. We consider $\det p(\xi')$ as in the Lemma 2.3. □

Lemma 3.4.

i) *Assume* $c_j \neq 0$*;* $2 \leq k \leq m\ \forall k \neq j$*; then* $c_j \phi_k^j \phi_j^k + \phi_j^1 \phi_1^j$ *is real.*

ii) *Assume* $c_j = 0\,\forall j$*, then* $\phi_1^k \phi_k^1$ *real* $\forall k$, $1 < k \leq m$.

Proof. i) We consider in $\det p(\xi')$ the coefficient of

$$\chi_2 \cdots \widehat{\chi}_k \cdots \chi_m$$

and we use also Lemma 3.1.

ii) Choose $\chi_k = 0$; the result is immediate by considering the coefficient of $\Pi_{l\neq k}\chi_l$ in $\det p(\xi')$. □

We need an effective symmetrization of submatrices of $a(\xi)$ in many cases.

Lemma 3.5. *Assume* $a(\xi)$ *uniformly diagonalizable;* $d(a) = m^2 - 3$*; assume the* $(m-1)$ *first elements of the last row of* $a(\xi)$ *have their real and imaginary parts which are linearly independent forms:* $\left(\xi_1^m, \eta_1^m, \ldots, \xi_m^{m-1}, \eta_{m-1}^m\right)$*, denote by* z^m *this set and by* z' *the complementary of* z^m *in* z*.*

Assume that two forms of the diagonal are identical in χ_l $(\chi_k = 0)$*.*

To simplify the notations, we assume: $c_1 = c_2 = 1$, $c_3 = \cdots = c_m = 0$*; the analogous case for* c_j *is stated in the same manner or by interchanging rows and columns, so:*

$$a(\xi) = \begin{pmatrix} \xi_0+\chi_1+\psi(z)+i\lambda_1(z) & & & & \phi_m^1 \\ & \xi_0+\chi_1+i\lambda_2(z) & & & \\ & & & \phi_q^p(z)_{p<q} & \\ & & \delta_j & & \\ & \phi_l^i(z)_{l<i} & & & \\ z_1^m & z_2^m & z_j^m & & \delta_m \end{pmatrix}$$

Then, by change of basis in $\mathbb{C}^m$ *we obtain for* $a(\xi)$ *the form*

$$a(\xi) = \begin{pmatrix} \xi_0+\chi_1'+\psi'(z)+i\lambda_1(z^m) & k_2^1\overline{\phi_1^2}(z')+\phi_2^1(z^m) & k_j^1\overline{\phi_j^1}(z')+\phi_j^1(z^m) & \phi_m'^1(z) \\ \phi_1^2(z')+\phi_1^2(z^m) & \xi_0+\chi_1'+i\lambda_2(z^m) & k_j^2\overline{\phi_j^2}(z')+\phi_j^2(z^m) & \phi_m^2(z) \\ & & & \\ \phi_1^j(z')+\phi_1^j(z^m) & \phi_2^j(z) & \xi_0+\chi_j & k_m^j\overline{z_j^m} \\ & & & \\ z_1^m & z_2^m & z_j^m & \xi_0+\chi_m \end{pmatrix}$$

where $\lambda_1(z^m) + \lambda_2(z^m) = 0$, $k_j^i > 0$, $k_j^i = k_k^i k_j^k$, $1 \leq i < k < j \leq m-1$*;* $\phi_m^1(z) - \frac{h_{12}}{h_2}\phi_m^2(z) = \phi_m'^1(z)$.

Proof. By Lemma 3.1, $\lambda_j = 0\,\forall j \neq 1, 2$; we have a real trace and $\lambda_1(z)+\lambda_2(z) = 0$.

Let $z_1^m = z_2^m = \cdots = z_{m-1}^m = 0$ in $a(\xi)$ and consider the $(m-1)\times(m-1)$ matrix $b(\xi)$ obtained by the restricted elements $\phi_j^i(\xi)$, $1 \leq i, j \leq m-1$; $b(\xi)$ is prehermitian by induction and it exists H (Lemma 1.4) such that:

$$bH = H^t\overline{b}\,. \tag{3.5}$$

We denote $h = (h_{ij})$, $h_{ii} = h_i > 0$.

We make this identity explicit. By considering the terms in χ, we easily obtain that:

$$h_{ij} = 0\,, \quad \forall i, j\,, \quad i \neq j \quad \text{except } h_{12}\,.$$

We make the element in second row and second column in (3.5) explicit:

$$\phi_1^2(z')h_{12} + 2ih_2\lambda_2(z') = \overline{h_{12}}\,\overline{\phi_1^2}(z')\,.$$

We deduce

$$h_2\lambda_2(z') + \operatorname{Im}\left(h_{12}\phi_1^2(z')\right) = 0\,, \tag{3.6}$$

$$\phi_1^2(z')\left(\frac{h_{12}}{h_2}\right)^2 + 2i\frac{h_{12}}{h_2}\lambda_2(z') = \frac{|\,h_{12}\,|^2}{h_2^2}\overline{\phi_1^2}(z')\,. \tag{3.7}$$

Make the element in jth row, kth column, $j > k > 2$ explicit:

$$\phi_k^j(z')h_k = h_j\overline{\phi_j^k}(z');$$

so: $\phi_j^k(z) = \frac{h_k}{h_j}\overline{\phi_k^j}(z') + \phi_j^k(z^m)$.

Make the element in the jth row, $j > 2$ and 2nd column explicit:

$$\phi_1^j(z')h_{12} + \phi_2^j(z')h_2 = h_j\overline{\phi_j^2}(z')$$

so: $\phi_j^2(z') = \frac{h_2}{h_i}\overline{\phi_2^j}(z') + \frac{\overline{h_{12}}}{h_j}\overline{\phi_1^j}(z')$.

Make the element in jth row, 1st column explicit:

$$\phi_j^1(z') = \frac{h_1}{hj}\overline{\phi_1^j}(z') + \frac{h_{12}}{h_j}\overline{\phi_2^j}(z')\,.$$

Make the element in the 2nd row, 1st column explicit:

$$\phi_1^2(z')h_1 + \overline{h_{12}}\,(\chi_1 + i\lambda_2(z')) = \overline{h_{12}}\left(\chi_1 + \psi(z') - i\lambda_1(z') + h_2\overline{\phi_2^1}(z')\right)\,;$$

then

$$\phi_2^1(z') = \frac{h_1}{h_2}\overline{\phi_1^2}(z') - \frac{h_{12}}{h_2}\psi(z')\,.$$

We denote by E_{12} the $m\times m$ matrix, all the elements of which are zero, except the diagonal element $= 1$ and the element in the 1st row, 2nd column which is equal to h_{12}. We transform the preceding matrix $a(\xi)$ by E_{12} and we obtain:

$$E_{12}^{-1}a(\xi)E_{12}\,.$$

We state this matrix is the announced matrix in the lemma. The element in the 1st row, 1st column is

$$\chi_1 + \psi'(z) - i\lambda_2(z') - \frac{h_{12}}{h_2}\phi_1^2(z') - i\lambda_1(z^m)\,,$$

we use (3.6) to obtain the result.

The element in the 1st row, 2nd column is

$$-2i\frac{h_{12}}{h_2}\lambda_2(z') - \left(\frac{h_{12}}{h_2}\right)^2 \phi_1^2(z') + \frac{h_1}{h_2}\overline{\phi_1^2}(z') + \text{terms in } (z^m)\,;$$

the formula (3.7) implies this term is equal to

$$\frac{h_1 h_2 - \mid h_{12} \mid^2}{(h_2)^2}\overline{\phi_1^2}(z') + \text{terms in } (z^m)\,;$$

let: $k_2^1 = \frac{h_1h_2-|h_{12}|^2}{(h_2)^2} \geq 0$; we obtain the result.

In the same manner, the element in the 1st row jth column ($j < m$) is

$$k_j^1\overline{\phi_1^j}(z') + \phi_j^1(z^m)$$

with: $k_j^1 = \frac{h_1h_2-|h_{12}|^2}{h_2h_j} > 0$.

The element in the 1st row mth column is

$$\phi_m^1(z) - \frac{h_{12}}{h_2}\phi_m^2(z) = \phi_m'^1(z)\,.$$

The element in the jth row, 2nd column is

$$\phi_j^2(z) + \frac{h_{12}}{h_2}\phi_1^1(z) = \phi_2'^j(z)\,,$$

and in the 2nd row, jth column:

$$\phi_j^2(z) = k_j^2\overline{\phi_2'^j}(z') + \phi_j^2(z^m)\,, \qquad k_j^2 = \frac{h_2}{h_j}\,.$$

In the jth row, kth column, $j > k > 2$, we have $\phi_k^j(z)$ and the one in the kth row jth column is:

$$k_j^k\overline{\phi_k^j}(z') + \phi_j^k(z^m)\,; \qquad k_j^k = \frac{h_k}{h_j}\,.$$

We easily verify the properties of the k_q^p and obtain the Lemma.

Now we consider the different forms of $a(\xi)$ along the positions of dependent forms; all the positions reduce to seven ones. We draw only the dependent linear forms:

$$A \begin{pmatrix} \phi_1^1 & & \\ \operatorname{Re}\phi_1^2 + i\operatorname{Im}\phi_1^2 & & \\ & & \end{pmatrix}, \qquad B \begin{pmatrix} \phi_1^1 & & \\ \operatorname{Re}\phi_1^2 + i\eta_1^2 & & \\ \operatorname{Re}\phi_1^3 + i\eta_1^3 & & \\ & & \end{pmatrix},$$

$$C \begin{pmatrix} \phi_1^1 & & \\ \operatorname{Re}\phi_1^2 + i\eta_1^2 & & \\ & \operatorname{Re}\phi_2^3 + i\eta_2^3 & \\ & & \end{pmatrix}, \qquad D \begin{pmatrix} \phi_1^1 & & \\ & & \\ & \operatorname{Re}\phi_2^3 + i\operatorname{Im}\phi_2^3 & \\ & & \end{pmatrix},$$

$$E \begin{pmatrix} \phi_1^1 & & \\ \operatorname{Re}\phi_1^2 + i\eta_1^2 & & \\ & & \\ & & \operatorname{Re}\phi_3^4 + i\eta_3^4 \end{pmatrix}, \qquad F \begin{pmatrix} \phi_1^1 & & \\ & & \\ & \operatorname{Re}\phi_2^3 + i\operatorname{Im}\phi_2^3 & \\ & \operatorname{Re}\phi_2^4 + i\operatorname{Im}\phi_2^4 & \end{pmatrix},$$

$$G \begin{pmatrix} \phi_1^1 & & \\ & & \\ & \operatorname{Re}\phi_2^3 + i\eta_2^3 & \\ & & \\ & & \operatorname{Re}\phi_4^5 + i\eta_4^5 \end{pmatrix}.$$

We study the different cases.

Case A. We distinguish some subcases.

Subcase A_1. $\quad c_2 \neq 0$.

By Lemmas 3.1, 3.2, we get immediately:

$$p(\xi') = \begin{pmatrix} c_2\chi_2 + \cdots + \psi(z) + i\lambda_1(z) & \phi_2^1(z) & & \phi_j^1(z) & \phi_m^1(z) \\ \operatorname{Re}\phi_1^2(z) + i\operatorname{Im}\phi_1^2(z) & \chi_2 + i\lambda_2(z) & & \phi_j^2(z) & \phi_m^2(z) \\ & & & & \\ z_1^j & & z_k^j & \chi_j & k_m^j\overline{z_j^m} \\ & & & & \\ z_1^m & & & z_j^m & 0 \end{pmatrix}$$

Let $z_1^m = z_2^m = \cdots = z_{m-1}^m = 0$, we consider the obtained $(m-1)\times(m-1)$ submatrix b we get by induction and by Lemmas 1.5 and 3.5:

$$\phi_j^1(z) = k_j^1\overline{\phi_1^j}(z') + \phi_j^1(z^m),$$
$$\phi_j^2(z) = k_j^2\overline{\phi_2^j}(z') + \phi_j^2(z^m), \quad j < m;$$

λ_1 depends only on z^m.

We have the convenient relations between k_j^i, $j \leq m-1$.

Using Lemma 3.4, we obtain:

$$\phi_j^1(z') = k_j^1\overline{\phi_1^j}(z'),$$
$$\phi_j^2(z') = k_j^2\overline{\phi_2^j}(z') \qquad 3 \le j \le m-1.$$

Let $\left(z_1^{m-1}, z_2^{m-1}, \dots, z_{m-2}^{m-1}, z_{m-1}^m\right) = z^{m-1} = 0$. We get:

$$\phi_m^1(z) = k_m^1\overline{z_1^m} + \phi_m^1\left(z^{m-1}\right),$$
$$\phi_m^2(z) = k_m^2\overline{z_2^m} + \phi_m^2\left(z^{m-1}\right).$$

Let $z'' = \{z'\}\,U\left\{\xi_1^m, \eta_1^m, \dots, \xi_{m-2}^m, \eta_{m-2}^m\right\}$

$$\phi_2^1(z) = k_2^1\overline{\phi_1^2}(z'') + \phi_2^1(\xi_{m-1}^m, \eta_{m-1}^m) > 0\,; \quad \lambda_2 \text{ depends only on } \xi_{m-1}^m, \eta_{m-1}^m\,.$$

We have: $k_m^1 = k_j^1 k_m^j, \qquad j \neq m-1$.

By Lemma 3.4, we obtain:

$$\phi_m^1(z) = k_m^1\overline{z_1^m},$$
$$\phi_m^2(z) = k_m^2\overline{z_2^m}.$$

Let $z_1^{m-2} = z_2^{m-2} = \cdots = z_{m-3}^{m-2} = z_{m-2}^{m-1} = z_{m-2}^m = 0$. We obtain:

$$\phi_2^1(z) = k_2^1\overline{\phi_1^2}(z)\,; \quad \lambda_2 = 0$$

and $k_m^{m-2} = k_{m-1}^{m-2}k_m^{m-1} > 0$.

We deduce a is prehermitian.

Subcase A_2. $\quad c_3 \neq 0 \quad (c_2 = 0)$; the proof is essentially similar to A_1.

Subcase A_3. $\quad c_4 \neq 0 \quad (c_2 = c_3 = 0)$; by interchanging the 4th and the 3rd row, and the 4th and the 3rd column, we reduce to A_2. In the same manner we state the result for $c_j \neq 0$, $j \geq 5$.

Subcase A_4. $\quad c_j = 0$, $\forall j$; let: $\xi_0 = \xi_0' + \chi_m$

$$\chi_j + \chi_m = \chi_j'\,, \quad 2 \le j \le m-2\,, \qquad \chi_{m-1} + \chi_m = 0\,.$$

Briefly

$$p(\xi') = \begin{pmatrix} \chi_m + \psi(z) + i\lambda_1(z) & & \phi_{m-1}^1 & \phi_m^1 \\ \phi_1^2(z) & \chi_2' & & \\ & & & \\ & & & \\ & & 0 & \\ z_1^{m-1} & & & \phi_m^{m-1}(z) \\ z_1^m & & z_{m-1}^m & \chi_m + i\lambda_m(z) \end{pmatrix}.$$

Let $z_1^{m-1} = z_2^{m-1} = \cdots = z_{m-2}^{m-1} = z_1^m = z_2^m = \cdots = z_{m-2}^m = 0$; we obtain:

$$\phi_m^{m-1} = k_m^{m-1}\overline{z_{m-1}^m} + \phi_m^{m-1}(z_1^{m-1}, \dots, z_{m-2}^{m1}, z_1^m, \dots, z_{m-2}^m),$$

$k_m^{m-1} > 0$.

We exchange the $(m-1)$th row for the mth one and the $(m-1)$th column for the mth one; we make the change of real variables:

$$\xi'^{m}_{m-1} + i\eta'^{m}_{m-1} \equiv k^{m-1}_{m}\left(\xi^{m}_{m-1} - i\eta^{m}_{m-1}\right) + \phi^{m-1}_{m}\left(\dots\right)$$

and we come back to the preceding case.

Case B.

Subcase B_1. $\quad c_2 \neq 0$

We recall that z is a basis of V ; $z = z'Uz^{^m}$; as in A_1, we get by Lemmas 3.1, 3.2, 1.4, 3.5:

$$p(\xi') = \begin{pmatrix} c_2\chi_2+\cdots+\psi(z)+i\lambda_1(z^m) & \alpha^1_2 & \alpha^1_3 & \alpha^1_j & \phi^1_m \\ \operatorname{Re}\phi^2_1(z)+i\eta^2_1 & \chi_2+i\lambda_2(z^m) & \alpha^2_3 & \alpha^2_j & \phi^2_m \\ \operatorname{Re}\phi^3_1(z)+i\eta^3_1 & z^3_2 & \chi_3 & & \\ & & & k^k_j\overline{z^j_k} & \\ & & & & \\ z^j_1 & z^j_2 & z^j_3 & & k^j_m\overline{z^m_j} \\ z^m_1 & & & z^m_j & 0 \end{pmatrix},$$

with $\alpha^k_j = k^k_j\overline{\phi^j_k}(z')+\phi^k_j(z^m)$, and the relations between the k^k_j, $1 \leq k < j \leq m-1$.

We remark that we can assume $\operatorname{Re}\phi^2_1(z)$ does not depend on η^3_1; otherwise we reduce to the case A; using Lemma 3.4, we obtain:

$$\begin{aligned} \phi^1_j(z) &= k^1_j\overline{\phi^j_1}(z), \\ \phi^2_j(z) &= k^2_j\overline{\phi^j_2}(z)\,, \quad 4 \leq j \leq m-1\,. \end{aligned}$$

Let $\xi^{m-1}_1 = \eta^{m-1}_1 = \cdots = \xi^{m-1}_{m-2} = \eta^{m-1}_{m-2} = \xi^m_{m-1} = \eta^m_{m-1} = 0$; as before, we obtain:

$$\begin{aligned} \phi^1_m(z) &= k^1_m\overline{z^m_1} + \phi^1_m\left(z^{m-1}\right) \\ \phi^2_m(z) &= k^2_m\overline{z^m_2} + \phi^2_m\left(z^{m-1}\right) \\ \phi^1_2(z) &= k^1_2\overline{\phi^2_1}\left(z''\right) + \phi^1_2(\xi^m_{m-1}, \eta^m_{m-1}) \\ \phi^1_3(z) &= k^1_3\overline{\phi^3_1}\left(z''\right) + \phi^1_3(\xi^m_{m-1}, \eta^m_{m-1}) \\ \phi^2_3(z) &= k^2_3\overline{z^3_2} + \phi^2_3(\xi^m_{m-1}, \eta^m_{m-1}) \end{aligned}$$

and relations among the k^i_j.

Consider in $\det p(\xi')$ the coefficient of $(\eta^3_1)^2\chi_4\cdots\chi_m$ (we replace 0 by χ_m, and χ_2 by 0); we obtain: $\lambda_2 = \lambda_1 = 0$.

By Lemma 3.4, considering the coefficient of $\chi_2\chi_3\dots\chi_{m-1}$ in $\det p(\xi')$ we obtain:

$$\phi^1_m(z) = k^1_m\overline{z^m_1}, \qquad \phi^2_m(\xi) = k^2_m\overline{z^m_2}\,.$$

We consider the coefficient of $\chi_2.\chi_5\dots\chi_m$ in $\det p(\xi')$.

$$c_{12}\det\begin{pmatrix} 0 & k_3^2\overline{z_2^3}+\phi_3^2(\xi_{m-1}^m,\eta_{m-1}^m) & k_4^2\overline{z_2^4} \\ z_2^3 & 0 & k_4^3\overline{z_3^4} \\ z_2^4 & z_3^4 & 0 \end{pmatrix}$$

$$+\det\begin{pmatrix} \phi_1(z) & k_3^1\overline{\phi_1^3}\,(z'')+\phi_3^1(\xi_{m-1}^m,\eta_{m-1}^m) & k_4^1\overline{z_1^4} \\ \phi_1^3(z) & 0 & k_4^3\overline{z_3^4} \\ z_1^4 & z_3^4 & 0 \end{pmatrix}$$

is real and: $\phi_3^2(\xi_{m-1}^m,\eta_{m-1}^m)=0$; $\phi_3^1(\xi_{m-1}^m,\eta_{m-1}^m)=0$.

We consider the coefficient of $\chi_3\chi_5\dots\chi_m$ and we obtain $\phi_2^1=k_2^1\overline{\phi_1^2}$ we have the relation between the k_j^i.

Subcase B_2. $c_3\neq 0$ $(c_2=0)$. By interchanging rows and columns, we reduce to the case B_1.

Subcase B_3. $c_4\neq 0$ $c_2=c_3=0$.

The calculus are essentially analogous to the ones of the case B_1.

The cases $c_j\neq 0$ reduce to the preceding ones and also the case $c_j=0$ $\forall j$.

Cases C, D, E and F.

For these cases we use the same methods as for the preceding ones. All the cases with dependent forms in the first $m-1$ rows reduce to the cases A, B, C, D, E, F.

Case G.

Subcase G_1. $c_2\neq 0$. We distinguish at first the case $m=5$.

We remark that $\operatorname{Re}\phi_4^5$ depends only on the independent form η_4^5. Otherwise, we can reduce to a preceding case by change of variables and by exchanging rows for columns.

Let $z_1^5=z_2^5=z_3^5=\eta_4^5=0$ and use Lemma 3.5 for the submatrix b obtained.

Let $z_1^2=\dots=z_1^5=0$ and use the evident analogous of Lemma 3.5 for the submatrix obtained with the last 4 rows and columns of a. Use also Lemmas 3.1, 3.2, 3.3, we get:

$$p(\xi')=\begin{pmatrix} \delta & k_2^1\overline{z_1^2}+\phi_2^1(z^5) & k_3^1\overline{z_1^3}+\phi_3^1(z^5) & k_4^1\overline{z_1^4}+\phi_4^1(z^5) & \phi_5^1 \\ z_1^2 & \chi_2+i\lambda_2\left(z_1^5\right) & k_3^2.\overline{\phi_2^3}(z'')+\phi_3^2\left(z_1^5\right) & k_4^2\overline{z_2^4}+\phi_4^2\left(z_1^5\right) & k_5^2\overline{z_2^5}+\phi_5^2(z_1) \\ z_1^3 & \phi_2^3(z) & \chi_3 & k_4^3\overline{z_3^4} & k_5^3\overline{z_3^5} \\ z_1^4 & z_2^4 & z_3^4 & \chi_4 & k_5^4(d-i)\eta_4^5 \\ z_1^5 & z_2^5 & z_3^5 & (d+i)\eta_4^5 & 0 \end{pmatrix}$$

with $\delta=c_2\chi_2+\dots+\psi(z)+i\lambda_1(z_1^5)$. We let:

$$z_1=(\xi_1^2,\eta_1^2,\xi_1^3,\eta_1^3,\xi_1^4,\eta_1^4,\xi_1^5,\eta_1^5)$$

and here: $z''=\left\{\text{all the independent variables except: }\xi_1^5,\eta_1^5\right\}$.

If $m>5$, let $z_1^m=\dots=z_{m-1}^m=0$ and use Lemma 3.5; let $z_1^2\dots z_1^m=0$ and use again Lemma 3.5.

As before, using also Lemmas 3.1, 3.2, 3.3, we get:

$$p(\xi') = \begin{pmatrix} \delta & k_2^1\overline{z_1^2}+\phi_2^1(z^m) & k_3^1\overline{z_1^3}(z)+\phi_3^1(z^m) & k_4^1\overline{z_1^4}+\phi_4^1(z^m) & & \phi_m^1 \\ z_1^2 & \chi_2+i\lambda_2(z_1^m) & k_3^2\overline{\phi_2^3}(z'')+\phi_3^2(z_1^m) & k_4^2\overline{z_2^4}+\phi_4^2(z_1^m) & & k_m^2\overline{z_2^m}+\phi_m^2(z_1) \\ z_1^3 & \phi_2^3(z) & \chi_3 & k_4^3\overline{z_3^4} & & k_m^3\overline{z_3^m} \\ z_1^4 & z_2^4 & z_3^4 & \chi_4 & \alpha_5^4 & k_m^4\overline{z_4^m} \\ z_1^5 & z_2^5 & z_3^5 & \operatorname{Re}\phi_4^5(z_4)+i\eta_4^5 & \chi_5 & k_m^5\overline{z_5^m} \\ z_1^m & z_2^m & z_3^m & z_4^m & z_5^m & 0 \end{pmatrix}$$

with $\delta = c_{12}\chi_2 + \cdots + \psi(z) + i\lambda_1(z_1^m)$, $\alpha_5^4 = k_5^4 \operatorname{Re}\phi_4^5(z_4) - i\eta_4^5$. We let $z_4 = \left(\eta_4^5, \xi_j^6, \eta_j^6, \ldots, \xi_j^m, \eta_j^m\right)$, $j \geq 4$, as before, if ϕ_4^5 depends on other coordinates, we reduce to the preceding case. In all the cases, we have the wished relation between the k_j^i, $(i,j) \neq (1,m)$.

Let now, in all the cases:
$z_1^2 = z_1^3 = \cdots = z_1^{m-1} = 0$; $z_3^m = z_4^m = \cdots = z_{m-1}^m = 0$;
$k_m^2(\xi_2^m - i\eta_2^m) + \phi_m^2\left(\xi_1^m, \eta_1^m\right) = 0$.

We consider the $(m-2)\times(m-2)$ submatrix obtained and we get by adaptation of Lemma 3.5:

$$\lambda_1 = \lambda_2 = 0\,; \quad \phi_j^2(z_1^m) = 0\,, \quad 4 \leq j \leq m-1\,; \quad \phi_3^2(z) = k_3^2\overline{\phi_2^3}(z).$$

In $\det p(\xi')$, consider the coefficient of $\chi_2 \ldots \widehat{\chi_j}\ \chi_m$, $3 \leq j \leq m-1$ (we have replaced χ_j by χ_m); we obtain: $\phi_j^1(z) = k_j^1\overline{\phi_1^j}(z)$.

If $m = 5$, let $\xi_1^4 = \eta_1^4 = \xi_2^4 = \eta_2^4 = \xi_3^4 = \eta_3^4 = \eta_4^5 = 0$; by Lemma 3.5, we obtain:

$$\phi_5^1(z) = k_5^1\overline{z_1^5} + \phi_5^1(\xi_1^4, \eta_1^4, \xi_2^4, \eta_2^4, \xi_3^4, \eta_3^4, \eta_4^5),$$
$$\phi_5^2(z) = k_5^2\overline{z_2^5} + \phi_5^2(z_1^4), \qquad \phi_2^1(z) = k_2^1\overline{z_1^2} + \phi_2^1(\eta_4^5)\,.$$

If $m = 6$, let

$$z_1^5 = z_2^5 = z_3^5 = 0\,; \quad z_5^6 = 0\,; \quad \eta_4^5 = z_4^6 = 0\,,$$

we obtain:

$$\phi_6^1(z) = k_6^1\overline{z_1^6} + \phi_6^1\left(z_1^5, z_2^5, z_3^5, \eta_4^5, z_4^6, z_5^6\right)$$
$$\phi_6^2(z) = k_6^2\overline{z_2^6} + \phi_6^2\left(z_1^5\right), \qquad \phi_2^1(z) = k_2^1\overline{z_1^2} + \phi_2^1\left(z_4^6, z_5^6\right)\,.$$

If $m > 6$, we easily obtain:

$$\phi_m^1(z) = k_m^1\overline{z_1^m} + \phi_m^1\left(z^{m-1}\right), \qquad \phi_m^2(z) = k_m^2\overline{z_2^m} + \phi_m^2\left(z_1^{m-1}\right)$$
$$\phi_2^1(z) = k_2^1\overline{z_1^2} + \phi_2^1\left(z_{m-1}^m\right)\,.$$

Finally, we consider in $\det p(\xi')$ the coefficient of $\chi_2\chi_3 \ldots \chi_{m-1}$, and we obtain: $\phi_m^1(z) = k_m^1\overline{z_1^m}$, $\phi_m^2(z) = k_m^2\overline{z_2^m}$.

We consider the coefficient of $\chi_3, \ldots, \chi_m$ and we obtain $\phi_2^1(z) = k_2^1\overline{z_1^2}$;

We have also $k_m^1 = k_2^1 k_m^2$ and the result.

Subcase G_2. $c_2 = 0$, $c_3 \neq 0$ This case reduces to the previous one.

Subcases G_k. $c_2 = c_3 = \cdots = \cdots = c_k = 0\,, \quad c_{k+1} \neq 0\,, \quad 3 \leq k \leq m-2\,.$

The cases G_3, G_4, G_5 are essentially analogous to G_1; the other cases reduce to the preceding ones $\quad G_m \cdot c_j = 0 \quad \forall j$.

The proof is essentially analogous to the one of G_1.

We verify that all the positions for the dependent forms reduce to the preceding ones. □

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